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Maurer's Homotopy Theory and Geometric Algebra for Even Δ-Matroids

Walter Wenzel*

LaBRI, Université de Bordeaux 1, 351, Cours de la Libération, F-33405 Talence Cedex, France

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The Tutte group of a matroid M is a certain abelian group which controls the representability of M. The representation theory of matroids and that of even Δ-matroids have much in common. This paper is devoted to the extension of the concept of the Tutte group to even Δ-matroids defined on sets of arbitrary cardinality. Similarly as in ordinary matroid theory the Tutte group can be defined in several possible ways in terms of generators and relations.

INTRODUCTION

There are matroids which cannot be coordinatized over some commutative ring or field. Hence, to develop and to study the geometric algebra of matroids it has become necessary to look for more appropriate algebraic invariants. In [DW1] we associated with any matroid M, defined on a finite set E, a certain abelian group $\mathbb{T}_M$ which in view of its close relations to Tutte's coordinatization theory as developed in [T] we have christened the Tutte group of M. Several properties of the Tutte group were extended to matroids defined on sets of arbitrary cardinality in [DW3].

In 1987 Bouchet introduced the concept of a Δ-matroid obtained by some weakening of the structure of a matroid (cf. [B1]), which is closely related to the concept of a metroid (see [BDH, DH]). Representability problems for Δ-matroids are studied in [B2, BD], and [W2, W3], and it turned out (cf. [B2, Section 4]) that for every field K a matroid M is representable over K if and only if it is representable over K as a

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Δ-matroid by some skew-symmetric matrix. This result suggested that it should be possible to extend the concept of the Tutte group to at least a large class of Δ-matroids. Indeed, in the present paper it is shown that for the ‘‘even Δ-matroids,’’ which are very important by studying skew-symmetric matrices, symplectic geometries (cf. [W2, W7]) and matchings in graphs (see [B3, KW]), we can define a Tutte group which controls representability problems in an analogous way as the Tutte group in ordinary matroid theory does. In [W6] a unified theory of the Tutte group of more general combinatorial geometries, which are related to the (W, P)-matroids as examined by Gelfand and Serganova in [GS1, GS2], is developed. The present paper is devoted to list those properties of the Tutte group which are important in the case of even Δ-matroids. Since in symplectic geometry even Δ-matroids defined on infinite sets are quite essential we allow the sets on which the Δ-matroids are defined to be infinite; however, by definition, all feasible sets will be finite.

Technically, the paper is organized as follows: In Section 1 we recall the basic properties of even Δ-matroids and extend Maurer’s homotopy theory as developed in [M] for ordinary matroids to even Δ-matroids. This has already been done in [W4] in the much more general and abstract framework of combinatorial $(W; P; U)$-geometries; however, in the case of even Δ-matroids most of the considerations become much simpler, and therefore it seems to be more than adequate to develop the extension of Maurer’s homotopy theory in this special and quite important case once more directly.

In Section 2 we show that the Tutte group $\mathbb{T}_M$ of an even Δ-matroid M may be defined by generators and relations in several possible ways. For this purpose we shall use the extension of Maurer’s homotopy theory as developed in Section 1. In the next section we study essentially canonical homomorphisms $\mathbb{T}_{M'} \to \mathbb{T}_M$ for a minor M' of an even Δ-matroid M as well as for its dual $M' = M^*$.

In Section 4 we compute the Tutte group of an even Δ-matroid M from the Tutte groups of its connected components as defined in [B4] and specify a particular subgroup $\mathbb{T}_M^{(0)}$ of $\mathbb{T}_M$. It is shown that the structure of $\mathbb{T}_M$ can be computed from the structure of $\mathbb{T}_M^{(0)}$ and certain invariants concerning the connected components of M. Finally, in the last section the relationship between Δ-matroids with coefficients as introduced in [W3] and their Tutte groups is studied. Δ-matroids with coefficients are some—more or less canonical—extension of matroids with coefficients as studied in [D], [DW2, DW4].

In a forthcoming paper (cf. [W7]) it will be shown that the particular subgroup $\mathbb{T}_M^{(0)}$ of $\mathbb{T}_M$ for an even Δ-matroid M controls which embeddings of M into symplectic vector spaces differ only by simple geometric transformations.

1. MAURER'S HOMOTOPY THEORY FOR EVEN Δ-MATROIDS

In the sequel we assume that E is a—not necessarily finite—set.

DEFINITION 1.1. Assume

$$\emptyset \neq \mathscr{F} \subseteq \mathscr{P}_{\text{fin}}(E) := \{F \subseteq E \mid \#F < \infty\}.$$

Then the pair $(E, \mathscr{F})$ is called a **Δ-matroid** if the following **symmetric exchange axiom** is satisfied:

SEA. For $F_1, F_2 \in \mathscr{F}$ and $e \in F_1 \Delta F_2 := (F_1 \cup F_2) \setminus (F_1 \cap F_2)$ there exists some $f \in F_1 \Delta F_2$ with $F_1 \Delta \{e, f\} \in \mathscr{F}$.

$\mathscr{F}$ is called the system of *free* (or feasible) subsets of the Δ-matroid.

Note that by definition the feasible subsets of a Δ-matroid are finite in any case.

EXAMPLE. Assume that K is a field and E is some spanning subset of the vector space K^m for some $m \geq 0$. If $\mathscr{F}_1$ denotes the vector bases of K^m which are contained in E, then by basic linear algebra $(E, \mathscr{F}_1)$ is a Δ-matroid. If $\mathscr{F}_2$ denotes the linear independent subsets of E then $(E, \mathscr{F}_2)$ is a Δ-matroid, too.

In this paper we shall only be concerned with even Δ-matroids.

DEFINITION 1.2. (i) A Δ-matroid $(E, \mathscr{F})$ is *even* if one has $\#F_1 \equiv \#F_2 \bmod 2$ for all $F_1, F_2 \in \mathscr{F}$.

(ii) A Δ-matroid $(E, \mathscr{F})$ satisfies the *strong exchange condition* (or strong exchange axiom) if for all $F_1, F_2 \in \mathscr{F}$ and $e \in F_1 \Delta F_2$ there exists some $f \in (F_1 \Delta F_2) \setminus \{e\}$ with $F_1 \Delta \{e, f\} \in \mathscr{F}$ and $F_2 \Delta \{e, f\} \in \mathscr{F}$.

Note that in the above example the Δ-matroid $(E, \mathscr{F}_1)$ is even, while the Δ-matroid $(E, \mathscr{F}_2)$ is not if $m \geq 1$. One has the following.

PROPOSITION 1.3. *Assume $(E, \mathscr{F})$ is a Δ-matroid. Then the following three statements are equivalent*:

(i) *The Δ-matroid $(E, \mathscr{F})$ is even.*

(ii) *For $F_1, F_2 \in \mathscr{F}$ and $e \in F_1 \Delta F_2$ there exists some $f \in (F_1 \Delta F_2) \setminus \{e\}$ with $F_1 \Delta \{e, f\} \in \mathscr{F}$.*

(iii) *The Δ-matroid $(E, \mathscr{F})$ satisfies the strong exchange axiom.*

Proof. This is Theorem 2 in [W1]. ∎

If all feasible sets of a Δ-matroid are equicardinal, then axiom SEA is equivalent to the exchange axiom for bases in matroids. Hence, the matroids defined on E are exactly the Δ-matroids defined on E with equicardinal feasible sets, and all these Δ-matroids are trivially even.

For $\mathscr{F} \subseteq \mathscr{P}_{\text{fin}}(E)$ and $I \in \mathscr{P}_{\text{fin}}(E)$ put

$$\mathscr{F}\Delta I := \{F\Delta I \mid F \in \mathscr{F}\}. \tag{1.1}$$

Then it is also clear that $(E, \mathscr{F})$ is a Δ-matroid if and only if $(E, \mathscr{F}\Delta I)$ is a Δ-matroid, and either none or both of these are even.

In the rest of this section we want to extend Maurer's homotopy theory as established for ordinary matroids in [M, Section 5] to even Δ-matroids. To this end we state the following.

DEFINITION 1.4. Assume $G = (V(G), \mathscr{K}(G))$ is a graph with $V(G)$ as its set of vertices and $\mathscr{K}(G)$ its set of edges.

(i) If $V' \subseteq V(G)$ and $\mathscr{K}' \subseteq \mathscr{K}(G) \cap \binom{V'}{2}$, then the graph $G' := (V', \mathscr{K}')$ is *isometric in G*, if for all $v_1, v_2 \in V'$ one has

$$d_{G'}(v_1, v_2) = d_G(v_1, v_2) := \begin{cases} \infty, & \text{if } v_1 \text{ and } v_2 \text{ lie in distinct} \\ & \text{connected components of } G \\ \min\{l \in \mathbb{N}_0 \mid \text{there exist } w_0, \dots, w_l \in V \text{ with} \\ \quad w_0 = v_1, w_l = v_2, \text{ and } \{w_{i-1}, w_i\} \in \mathscr{K} \\ \quad \text{for } 1 \le i \le l\}, & \text{otherwise}. \end{cases}$$

(ii) A reentrant[1] path $(v_0, \dots, v_l)$ is called *isometric in G*, if $v_i = v_j$ only for $i \equiv j \bmod l$ and the cycle $G' = (V', \mathscr{K}')$ with $V' := \{v_0, \dots, v_l\}$, $\mathscr{K}' := \{\{v_i, v_j\} \mid j \equiv i + 1 \bmod l\}$ is isometric in G.

(iii) The graph G satisfies the *Maurer condition*, if for any two isometric reentrant paths $(v_0, \dots, v_l)$, $(v'_0, \dots, v'_l)$ in G of length $l \ge 5$ with $v_i = v'_i$ for $0 \le i \le l-2$ one has either $v'_{l-1} = v_{l-1}$ or $d_G(v_{l-1}, v'_{l-1}) = 1$.

We have the following quite useful proposition.

PROPOSITION 1.5. *Assume $G = (V(G), \mathscr{K}(G))$ is a graph which satisfies the Maurer condition. If H is an isometric subgraph in G, then H satisfies the Maurer condition, too.*

Proof. Assume $Z = (v_0, \dots, v_l)$ and $Z' = (v'_0, \dots, v'_l)$ are isometric reentrant paths in H of length $l \ge 5$ and $v_i = v'_i$ for $0 \le i \le l-2$. Since

[1] "Reentrant" means that $v_0 = v_l$.

H is isometric in G, the paths Z and Z' are isometric in G, too. This implies either $v'_{l-1} = v_{l-1}$ or $d_H(v_{l-1}, v'_{l-1}) = d_G(v_{l-1}, v'_{l-1}) = 1$. ∎

In what follows we assume that $M = (E, \mathscr{F})$ is an even Δ-matroid and $F_0 \in \mathscr{F}$ is fixed. (We may choose $F_0 = \emptyset$, if $\emptyset \in \mathscr{F}$.)

DEFINITION 1.6. (i) Let G_M denote the graph with vertex set

$$V_0 := \{F \in \mathscr{P}_{\text{fin}}(E) \mid \#F \equiv \#F_0 \bmod 2\} \tag{1.2a}$$

and edge set

$$\mathscr{K}_0 := \{\{F_1, F_2\} \subseteq V_0 \mid \#(F_1 \Delta F_2) = 2\}. \tag{1.2b}$$

(ii) The *base graph* Γ_M of M is the induced subgraph of G_M on the set $\mathscr{F}$ of the vertices.

Note that G_M does, of course, not depend on the chosen set F_0, because M is even. For simplicity we denote the metric d_{G_M} on V_0 by d. We get

$$d(F_1, F_2) = \tfrac{1}{2} \cdot \#(F_1 \Delta F_2) \qquad \text{for all } F_1, F_2 \in V_0. \tag{1.3}$$

Definition 1.6 (ii) recovers, of course, the definition of the base graph of an ordinary matroid. We have the following.

PROPOSITION 1.7. *The base graph Γ_M is isometric in G_M. In particular, Γ_M is connected.*

Proof. This result follows from the characterization of even Δ-matroids given in Proposition 1.3 (ii):
If $F_1, F_2 \in \mathscr{F}$ and $e \in F_1 \Delta F_2, f \in (F_1 \Delta F_2) \setminus \{e\}$ satisfy $F'_1 := F_1 \Delta \{e, f\} \in \mathscr{F}$ then one has $d(F_1, F'_1) = 1$ and $d(F_1, F_2) = d(F'_1, F_2) + 1$; thus the assertion follows by induction. ∎

Now we are able to prove the following basic proposition.

PROPOSITION 1.8. *The graphs G_M and Γ_M satisfy the Maurer condition.*

Proof. By Propositions 1.5 and 1.7 it suffices to show that G_M satisfies the Maurer condition. Assume $l \geq 5$ and $Z = (A_0, \ldots, A_l)$, $Z' = (A'_0, \ldots, A'_l)$ are isometric reentrant paths in G_M with $A_i = A'_i$ for $0 \leq i \leq l - 2$. We must prove $A_{l-1} = A'_{l-1}$ or $\#(A_{l-1} \Delta A'_{l-1}) = 2$.
Since the map $\sigma : \mathscr{P}_{\text{fin}}(E) \to \mathscr{P}_{\text{fin}}(E)$, given by $\sigma(A) := A \Delta A_0$ satisfies $\#(\sigma(A) \Delta \sigma(B)) = \#(A \Delta B)$ for all $A, B \in \mathscr{P}_{\text{fin}}(E)$, we may assume $A_0 = \emptyset$.

Now we consider two cases:

Case I. l is even. In this case put $h := \frac{1}{2} \cdot l$. Since Z is isometric in G_M, we have $\# A_h = 2 \cdot h = l$, $A_i \subseteq A_h$ for all i with $0 \le i \le l$ and $\#(A_{h-1} \Delta A_{l-1}) = 2 \cdot h$. This is possible only if $A_{l-1} = A_h \setminus A_{h-1}$. Similarly, we have $A'_{l-1} = A_h \setminus A_{h-1}$. Thus we have $A_{l-1} = A'_{l-1}$ in this case.

Case II. l is odd. Now put $h := \frac{1}{2} \cdot (l-1)$. Then one has $\# A_h = 2 \cdot h$, and there exist $c, d, c', d' \in E$ with

$$A_{l-1} = \{c, d\}, \qquad A'_{l-1} = \{c', d'\}.$$

All we need to show is that $A_{l-1} \cap A'_{l-1} \neq \emptyset$. Otherwise we would have $A_{l-2} = \{c, d, c', d'\}$, because $l \ge 5$ implies $\# A_{l-2} = 4$. In view of

$$h - 1 = d(A_h, A_{l-2}) < d(A_h, A_{l-1}) = h$$

this means $A'_{l-1} = A_{l-2} \setminus A_{l-1} \subseteq A_h$ and, thus,

$$d(A'_{l-1}, A_h) = \tfrac{1}{2} \cdot \#(A_h \setminus A'_{l-1}) = h - 1,$$

which is impossible, because the path Z' is isometric in G_M. ∎

Remark. Proposition 1.8 recovers the fact that the base graph of a matroid satisfies the Maurer condition, too—a fact which was already proved in [DW4, Section 8, Satz 3].

For a graph G with $V := V(G)$ as its set of vertices and $\mathscr{K} := \mathscr{K}(G)$ its set of edges put

$$V^{(2)} := \big\{(u, v) \in V^2 \,|\, \{u, v\} \in \mathscr{K}\big\}, \tag{1.4}$$

$$C_0(G) := \mathbb{Z}[V], \tag{1.5}$$

$$C_1(G) := \mathbb{Z}[V^{(2)}] / \langle \{(u, v) + (v, u) \,|\, \{u, v\} \in \mathscr{K}\} \rangle; \tag{1.6}$$

thus $C_0(G)$ is the free abelian group generated by the vertices $v \in V$, while $C_1(G)$ is the quotient of the free abelian group generated by the elements $(u, v) \in V^{(2)}$ modulo the subgroup generated by all $(u, v) + (v, u)$ for $\{u, v\} \in \mathscr{K}$. For $(u, v) \in V^{(2)}$ let $\overline{(u, v)}$ denote the corresponding coset in $C_1(G)$. Then one has

$$C_1(G) = \left\{ \sum_{i=1}^{l} \overline{(u_i, v_i)} \,|\, l \ge 0, \{u_i, v_i\} \in \mathscr{K} \text{ for } 1 \le i \le l \right\} \cong \mathbb{Z}^{\mathscr{K}}_{\text{fin}}, \tag{1.6a}$$

where for a set S the set $\mathbb{Z}^S_{\text{fin}}$ consists of all maps $f : S \to \mathbb{Z}$ with $\# f^{-1}(\mathbb{Z} \setminus \{0\}) < \infty$. Moreover, we have a well-defined homomorphism

$\partial : C_1(G) \to C_0(G)$ given by

$$\partial\left(\sum_{i=1}^{l} \overline{(u_i, v_i)}\right) := \sum_{i=1}^{l} v_i - \sum_{i=1}^{l} u_i. \tag{1.7}$$

Finally, put

$$H_1(G) := \operatorname{Ker} \partial, \tag{1.8}$$

$$H_0(G) := \operatorname{Coker} \partial. \tag{1.9}$$

Then the following sequence of abelian groups is exact:

$$0 \hookrightarrow H_1(G) \hookrightarrow C_1(G) \xrightarrow{\partial} C_0(G) \twoheadrightarrow H_0(G) \twoheadrightarrow 0.$$

The group $H_0(G)$ is canonically isomorphic to $\mathbb{Z}_{\text{fin}}^{\mathscr{C}}$ where $\mathscr{C}$ denotes the set of connected components of G. In particular, one has $H_0(G) \cong \mathbb{Z}$ if and only if G is connected.

For a reentrant path $Z = (v_0, \ldots, v_l)$ in G of length l put

$$c_Z := \sum_{i \bmod l} \overline{(v_{i-1}, v_i)}. \tag{1.10}$$

Then one has $c_Z \in H_1(G)$.

DEFINITION 1.9. A reentrant path Z in G of length l is called *irreducible* if

$$c_Z \notin \langle \{ c_{Z'} \mid Z' \text{ is a reentrant path in } G \text{ of length } l' \text{ for some } l' < l \} \rangle.$$

Now we have the following simple lemma.

LEMMA 1.10 (cf. also [DW4, Section 8, Satz 1]). (i) *Every reentrant irreducible path in G is isometric in G.*

(ii) *The group $H_1(G)$ satisfies*

$$H_1(G) = \langle \{ c_Z \mid Z \textit{ is a reentrant irreducible path in } G \} \rangle.$$

Proof. (i) Assume $Z_0 = (v_0, \ldots, v_l)$ is a reentrant path in G of length l, which is not isometric in G. Then one has $l \geq 4$, and by symmetry we may assume that there exists some ν with $2 \leq \nu \leq l-2$ and $\mu := d_G(v_0, v_\nu) < \min\{\nu, l-\nu\}$. (It may be that $\mu = 0$; that is, $v_\nu = v_0$.) Choose some elements $w_0, \ldots, w_\mu \in V$ with $w_0 = v_0$, $w_\mu = v_\nu$, and $\{w_{i-1}, w_i\} \in \mathscr{K}$ for $1 \leq i \leq \mu$. Put $Z_1 := (v_0, \ldots, v_\nu, w_{\mu-1}, \ldots, w_0)$ and $Z_2 := (w_0, \ldots, w_\mu, v_{\nu+1}, \ldots, v_l)$. Then one has $c_{Z_0} = c_{Z_1} + c_{Z_2}$, which means that Z_0 is not irreducible.

(ii) The group $H_1(G)$ is generated by all sums $\sum_{i=1}^{l}\overline{(v_{i-1}, v_i)}$, for which $l \geq 2$, $v_0, \ldots, v_l \in V$, $\{v_{i-1}, v_i\} \in \mathscr{K}$ for $1 \leq i \leq l$, and $v_0 = v_l$. Thus we have to show that for every reentrant path $Z_0 = (v_0, \ldots, v_l)$ in G of length $l \geq 2$ the element c_{Z_0} is a linear combination of certain c_Z, where each Z is irreducible. This is easily seen by induction on l: In case $l = 2$ one has $c_{Z_0} = \overline{(v_0, v_1)} + \overline{(v_1, v_0)} = 0$, and there is nothing to prove.

Now assume $l \geq 3$. If Z_0 is itself irreducible, then the claim is obvious. But otherwise the assertion follows trivially from Definition 1.9 and the induction hypothesis. ∎

In order to study the group $H_1(\Gamma_M)$ for the base graph of M more thoroughly, we propose the following.

Definition 1.11. An isometric reentrant path $(A_0, A_1, A_2, A_3, A_0)$ in Γ_M of length 4 is called *strongly degenerate*, if there does not exist any $F \in \mathscr{F}$ with $d(F, A_i) = 1$ for $0 \leq i \leq 3$.

Now we are able to state and to prove the following result, which is an algebraic reformulation and generalization of Maurer's homotopy theorem for the base graph of an ordinary matroid (cf. [M, Theorem 5.1]).

Theorem 1.12. *The group $H_1(\Gamma_M)$ for the base graph Γ_M of the even Δ-matroid $M = (E, \mathscr{F})$ satisfies*

$$H_1(\Gamma_M) = \langle \{c_Z \mid Z \textit{ is a reentrant path in } \Gamma_M \textit{ of length } 3 \textit{ or}$$
$$\textit{a strongly degenerate path in } \Gamma_M \textit{ of length } 4\}\rangle.$$

Proof. By Lemma 1.10 (ii) it suffices to show that every irreducible reentrant path Z_0 of length $l \geq 4$ in Γ_M has length 4 and is strongly degenerate.

If $Z_0 = (A_0, A_1, A_2, A_3, A_0)$ is irreducible but not strongly degenerate, then choose some $F \in \mathscr{F}$ with $d(F, A_i) = 1$ for $0 \leq i \leq 3$ and put $Z_i := (F, A_{i-1}, A_i, F)$ for $i \bmod 4$. Then one gets $c_{Z_0} = c_{Z_1} + c_{Z_2} + c_{Z_3} + c_{Z_4}$, which contradicts the fact that Z_0 is irreducible.

Now assume $l \geq 5$ and $Z_0 = (A_0, \ldots, A_l)$ is an irreducible reentrant path in Γ_M of length l. Lemma 1.10 (i) implies that Z_0 is isometric in Γ_M, and thus we have $d(A_{l-2}, A_l) = 2$. Since M satisfies the strong exchange axiom (cf. Proposition 1.3), there exist $F_1, F_2 \in \mathscr{F}$ with $d(F_1, F_2) = 2$ and $d(F_i, A_j) = 1$ for $i \in \{1, 2\}$ and $j \in \{l-2, l\}$. By Proposition 1.8 the graph Γ_M satisfies the Maurer condition, and thus it follows that at least one of the two reentrant paths $Z' = (A_0, \ldots, A_{l-2}, F_1, A_l)$ and $Z'' = (A_0, \ldots, A_{l-2}, F_2, A_l)$, say Z', is not isometric in Γ_M. Therefore by Lemma 1.10 (i) there exist $h \in \mathbb{N}_0$ and reentrant paths Z_i of length $l_i < l$ for

$1 \le i \le h$ such that

$$c_{Z'} = \sum_{i=1}^{h} c_{Z_i}.$$

Moreover, $Z_{h+1} := (A_l, F_1, A_{l-2}, A_{l-1}, A_l)$ is a—not necessarily isometric—reentrant path of length 4, and one has

$$c_{Z_0} = c_{Z'} + c_{Z_{h+1}} = \sum_{i=1}^{h+1} c_{Z_i},$$

contradicting again the fact that Z_0 is irreducible. ∎

2. VARIOUS WAYS TO DEFINE THE TUTTE GROUP OF AN EVEN Δ-MATROID

In the sequel we assume that $M = (E, \mathscr{F})$ is an even Δ-matroid. As above let $\Gamma_M = (\mathscr{F}, \mathscr{K})$ denote the base graph of M, where

$$\mathscr{K} = \big\{\{F_1, F_2\} \subseteq \mathscr{F} \,|\, \#(F_1 \Delta F_2) = 2\big\}$$

and put $\mathscr{F}^{(2)} := \{(F_1, F_2) \in \mathscr{F}^2 \,|\, \#(F_1 \Delta F_2) = 2\}$.

DEFINITION 2.1. (i) A double pair $\Lambda = ((A, B), (A', B')) \in (\mathscr{F}^{(2)})^2$ is called a *couple in the base graph* Γ_M, if $\#\{A, B, A', B'\} = 4$ and there exist $a_1, a_2 \in A \Delta A'$ with $a_1 \neq a_2$ and

$$B = A\Delta\{a_1, a_2\}, \qquad B' = A'\Delta\{a_1, a_2\}.$$

(ii) A couple $\Lambda = ((A, A\Delta\{a_1, a_2\}), (A', A'\Delta\{a_1, a_2\}))$ in Γ_M is called *degenerate*, if there exists some $i \in \{1, 2\}$ such that for all $a \in (A \Delta A') \setminus \{a_1, a_2\}$ one has

$$A\Delta\{a_i, a\} \notin \mathscr{F} \qquad \text{or} \qquad A'\Delta\{a_i, a\} \notin \mathscr{F}.$$

EXAMPLES. (i) Assume $E = \{1, 2, 3, 4, 5, 6\}$ and $\mathscr{F} = \{F \subseteq E \,|\, \#F \equiv 0 \bmod 2\}$. Then $((\emptyset, \{1, 2\}), (E, E \setminus \{1, 2\}))$ is a couple in Γ_M which is not degenerate.

(ii) Assume $E = \{1, 2, 3, 4\}$ and $\mathscr{F} = \{F \subseteq E \,|\, \#F = 2,\ F \neq \{2, 3\}\}$. Then $((\{1, 2\}, \{1, 3\}), (\{3, 4\}, \{2, 4\}))$ is a degenerate couple in Γ_M.

(iii) If $(A_1, A_2, A_3, A_4, A_1)$ is a strongly degenerate path in Γ_M of length 4, then $((A_1, A_2), (A_3, A_4))$ is a degenerate couple in Γ_M.

(iv) Assume that M is a matroid, and $\Lambda = ((A, B), (A', B')) \in (\mathscr{F}^{(2)})^2$ is a couple in $\Gamma := \Gamma_M$. Let H or H' denote the uniquely determined hyperplane containing $A \cap B$ or $A' \cap B'$, respectively, and let C or C' denote the uniquely determined circuit contained in $A \cup B$ or $A' \cup B'$, respectively. Then one has

$$C' \setminus H = \{ b \in E \mid (A \cap B) \cup \{b\} \in \mathscr{F} \text{ and } (A' \cup B') \setminus \{b\} \in \mathscr{F} \},$$

$$C \setminus H' = \{ b \in E \mid (A' \cap B') \cup \{b\} \in \mathscr{F} \text{ and } (A \cup B) \setminus \{b\} \in \mathscr{F} \},$$

and therefore in any case $A \Delta B \subseteq C' \setminus H$ and $A \Delta B \subseteq C \setminus H'$. In particular, Definition 2.1 (ii) means that the couple Λ is degenerate if and only if $\#(C' \setminus H) = 2$ or $\#(C \setminus H') = 2$. Moreover, in the special case $d_\Gamma(A, A') = d_\Gamma(B, B') = 2$ the couple Λ is degenerate if and only if there exists at most one $F \in \mathscr{F}$ with $d_\Gamma(A, F) = d_\Gamma(A', F) = d_\Gamma(B, F) = d_\Gamma(B', F) = 1$.

Definition 2.2. Let $\mathbb{F}_M^{\mathscr{F}}$ denote the (multiplicatively written) free abelian group generated by the symbols ε and $X_{(e_1, \dots, e_m)}$ for $\{e_1, \dots, e_m\} \in \mathscr{F}$, $e_i \neq e_j$ for $i \neq j$, and let $\mathbb{K}_M^{\mathscr{F}}$ denote the subgroup of $\mathbb{F}_M^{\mathscr{F}}$ generated by ε^2 and all elements of the forms

$$\varepsilon \cdot X_{(e_{\tau(1)}, \dots, e_{\tau(m)})} \cdot X^{-1}_{(e_1, \dots, e_m)}$$

for $\{e_1, \dots, e_m\} \in \mathscr{F}$, τ an odd permutation in Σ_m,

$$X_{(e_1, \dots, e_k, a)} \cdot X^{-1}_{(e_1, \dots, e_k, b)} \cdot X_{(f_1, \dots, f_l, b)} \cdot X^{-1}_{(f_1, \dots, f_l, a)} \text{ if}$$

$$((\{e_1, \dots, e_k, a\}, \{e_1, \dots, e_k, b\}), (\{f_1, \dots, f_l, b\}, \{f_1, \dots, f_l, a\}))$$

is a degenerate couple in Γ_M, and

$$X_{(e_1, \dots, e_k, a, b)} \cdot X^{-1}_{(e_1, \dots, e_k)} \cdot X_{(f_1, \dots, f_l)} \cdot X^{-1}_{(f_1, \dots, f_l, a, b)} \text{ if}$$

$$((\{e_1, \dots, e_k, a, b\}, \{e_1, \dots, e_k\}), (\{f_1, \dots, f_l\}, \{f_1, \dots, f_l, a, b\}))$$

is a degenerate couple in Γ_M.

Then the group $\mathbb{T}_M^{\mathscr{F}}$ is defined by

$$\mathbb{T}_M^{\mathscr{F}} := \mathbb{F}_M^{\mathscr{F}} / \mathbb{K}_M^{\mathscr{F}}. \tag{2.1}$$

If $\mu : \mathbb{F}_M^{\mathscr{F}} \twoheadrightarrow \mathbb{T}_M^{\mathscr{F}}$ is the canonical epimorphism, then we put

$$\varepsilon_M^{\mathscr{F}} := \mu(\varepsilon), \tag{2.1a}$$

$$T_{(e_1, \dots, e_m)} := \mu\big(X_{(e_1, \dots, e_m)}\big) \qquad \text{for } \{e_1, \dots, e_m\} \in \mathscr{F}. \tag{2.1b}$$

To define groups which are related to $\mathbb{T}_M^{\mathscr{F}}$ we have to fix some total order "$\leq$" on E. However, as it will turn out, the groups which will be defined do not depend on this special chosen order up to isomorphism.

Definition 2.3. Let $\mathbb{F}_M^{[\mathscr{F}]}$ denote the free abelian group generated by the symbols ε and X_B for $B \in \mathscr{F}$. For $A, B \in \mathscr{F}$ with $\#(A \Delta B) = 2$,

$A \Delta B = \{a, b\}$, $a < b$, put

$$\varepsilon(\{A, B\}) := \varepsilon^{\#\{e \in A \cap B \mid a < e < b\}}. \tag{2.2}$$

Let $\mathbb{K}_M^{[\mathscr{F}]}$ denote the subgroup of $\mathbb{F}_M^{[\mathscr{F}]}$ generated by ε^2 and all elements of the form

$$X_A \cdot X_B^{-1} \cdot \varepsilon(\{A, B\}) \cdot X_{A'} \cdot X_{B'}^{-1} \cdot \varepsilon(\{A', B'\})$$

for a degenerate couple $\Lambda = ((A, B), (A', B'))$ in Γ_M. Then the group $\mathbb{T}_M^{[\mathscr{F}]}$ is defined by

$$\mathbb{T}_M^{[\mathscr{F}]} := \mathbb{F}_M^{[\mathscr{F}]} / \mathbb{K}_M^{[\mathscr{F}]}. \tag{2.3}$$

If $\nu : \mathbb{F}_M^{[\mathscr{F}]} \twoheadrightarrow \mathbb{T}_M^{[\mathscr{F}]}$ is the canonical epimorphism, then we put

$$\varepsilon_M^{[\mathscr{F}]} := \nu(\varepsilon), \tag{2.3a}$$

$$T_B := \nu(X_B) \qquad \text{for } B \in \mathscr{F}, \tag{2.3b}$$

$$\varepsilon_M(\{A, B\}) := \nu(\varepsilon(\{A, B\})) \qquad \text{for } (A, B) \in \mathscr{F}^{(2)}. \tag{2.3c}$$

Let $\mathscr{D}$ denote the set of triangles in the base graph Γ_M; that is,

$$\mathscr{D} := \left\{\{A_1, A_2, A_3\} \subseteq \mathscr{F} \mid d\left(A_i, A_j\right) = 1 \text{ for } 1 \le i < j \le 3\right\}. \tag{2.4}$$

We have the following simple but rather useful lemma.

LEMMA 2.4. *A subset $\{A_1, A_2, A_3\} \subseteq \mathscr{F}$ lies in $\mathscr{D}$ if and only if there exist $A \in \mathscr{P}(E)$ and pairwise distinct $a_1, a_2, a_3 \in E$ with $A_i = A \Delta \{a_i\}$ for $1 \le i \le 3$. In this case the set A and the elements a_1, a_2, a_3 are uniquely determined by A_1, A_2, A_3.*

Proof. It is clear that $\{A_1, A_2, A_3\} \in \mathscr{D}$ holds if there exist $A \in \mathscr{P}(E)$ and pairwise distinct $a_1, a_2, a_3 \in E$ with $A_i = A \Delta \{a_i\}$ for $1 \le i \le 3$.

Now assume that vice versa $\{A_1, A_2, A_3\} \in \mathscr{D}$. Then there exist $a, b, c, d \in E$ with $a \ne b$ and $c \ne d$ such that $A_2 = A_1 \Delta \{a, b\}$ and $A_3 = A_1 \Delta \{c, d\}$. Since $\#(\{a, b\} \Delta \{c, d\}) = \#(A_2 \Delta A_3) = 2$, one has $\#(\{a, b\} \cap \{c, d\}) = 1$, say $a = c$ and $a \ne b \ne d \ne a$. Thus for $A := A_1 \Delta \{a\}$, we obtain

$$A_1 = A \Delta \{a\}, \qquad A_2 = A \Delta \{b\}, \qquad A_3 = A \Delta \{d\}.$$

If also $A_1 = A' \Delta \{a'\}$, $A_2 = A' \Delta \{b'\}$, $A_3 = A' \Delta \{d'\}$ for some $A' \subseteq E$ and pairwise distinct $a', b', d' \in E$, then one has

$$A \Delta A' = \{a\} \Delta \{a'\} = \{b\} \Delta \{b'\} = \{d\} \Delta \{d'\}.$$

Since a, b, d are pairwise distinct, this is possible only for $a = a'$, $b = b'$, $d = d'$, $A = A'$. ∎

Now put

$$\mathscr{D}_1 := \{\{A\Delta\{a_1\}, A\Delta\{a_2\}, A\Delta\{a_3\}\} \in \mathscr{D} \mid A \subseteq E, a_1 < a_2 < a_3, a_2 \notin A\}, \tag{2.4a}$$

$$\mathscr{D}_2 := \{\{A\Delta\{a_1\}, A\Delta\{a_2\}, A\Delta\{a_3\}\} \in \mathscr{D} \mid A \subseteq E, a_1 < a_2 < a_3, a_2 \in A\}. \tag{2.4b}$$

By Lemma 2.4 it is trivial that $\mathscr{D} = \mathscr{D}_1 \dot{\cup} \mathscr{D}_2$.

Note that this partition depends in general on the chosen total order "$\leq$" on E. If $\{A_1, A_2, A_3\} \in \mathscr{D}_i$ for some $i \in \{1, 2\}$, then we call $\{A_1, A_2, A_3\}$ a triangle of the ith kind in Γ_M.

DEFINITION 2.5. Let $\mathbb{F}_M$ denote the free abelian group generated by the symbols ε and $X_{A,B}$ for $(A, B) \in \mathscr{F}^{(2)}$. Let $\mathbb{K}_M$ denote the subgroup of $\mathbb{F}_M$ generated by ε^2 and all elements of the form

$$\begin{aligned} &X_{A,B} \cdot X_{B,A} && \text{for } (A, B) \in \mathscr{F}^{(2)}, \\ &\varepsilon^{i-1} \cdot X_{A_1, A_2} \cdot X_{A_2, A_3} \cdot X_{A_3, A_1} && \text{for } \{A_1, A_2, A_3\} \in \mathscr{D}_i, i \in \{1, 2\}, \end{aligned}$$

and

$$X_{A,B} \cdot X_{A',B'} \qquad \text{for a degenerate couple } ((A, B), (A', B')) \text{ in } \Gamma_M.$$

Then the *Tutte group* $\mathbb{T}_M$ of M is defined by

$$\mathbb{T}_M := \mathbb{F}_M / \mathbb{K}_M. \tag{2.5}$$

If $\nu_0 : \mathbb{F}_M \twoheadrightarrow \mathbb{T}_M$ is the canonical epimorphism, then we put

$$\varepsilon_M := \nu_0(\varepsilon), \tag{2.5a}$$

$$T_{A,B} := \nu_0(X_{A,B}) \qquad \text{for } (A, B) \in \mathscr{F}^{(2)}. \tag{2.5b}$$

Remark. If M is a matroid of rank m—that is, one has $\# B = m$ for all $B \in \mathscr{F}$—then (2.4a) and (2.4b) read

$$\mathscr{D}_1 = \left\{\{A \cup \{a_1\}, A \cup \{a_2\}, A \cup \{a_3\}\} \subseteq \mathscr{F} \mid A \in \binom{E}{m-1},\right.$$
$$\left. A \cup \{a_1, a_2, a_3\} \in \binom{E}{m+2}\right\}, \tag{2.4a$'$}$$

$$\mathscr{D}_2 = \left\{ \{A \setminus \{a_1\}, A \setminus \{a_2\}, A \setminus \{a_3\}\} \subseteq \mathscr{F} \mid A \in \binom{E}{m+1}, \right.$$

$$\left. A \setminus \{a_1, a_2, a_3\} \in \binom{E}{m-2} \right\}. \quad (2.4\mathrm{b}')$$

In particular, for matroids the partition $\mathscr{D} = \mathscr{D}_1 \dot{\cup} \mathscr{D}_2$ does not depend on some fixed total order "$\leq$" on E.

Note that Definition 2.5 is formally different from [DW1, Definition 1.1], where it was not explicitly required that $X_{A_1, A_2} \cdot X_{B_1, B_2} \in \mathbb{K}_M$ holds for all degenerate couples $((A_1, A_2), (B_1, B_2))$ in Γ_M, but only for those which satisfy $d(A_1, B_1) = d(A_2, B_2) = 2$. However, for a degenerate couple $((A_1, A_2), (B_1, B_2))$ in Γ_M with $h := d(A_1, B_1) = d(A_2, B_2) \geq 3$ we conclude by induction on h that $X_{A_1, A_2} \cdot X_{B_1, B_2}$ lies in the subgroup of $\mathbb{F}_M$ generated by all products $X_{A, B} \cdot X_{B, A}$ for $(A, B) \in \mathscr{F}^{(2)}$ and all products $X_{A'_1, A'_2} \cdot X_{B'_1, B'_2}$ for which $((A'_1, A'_2), (B'_1, B'_2))$ is a degenerate couple in Γ_M with $d(A'_1, B'_1) = d(A'_2, B'_2) = 2$:

Let H denote the uniquely determined hyperplane containing $A_1 \cap A_2$ and C the uniquely determined circuit contained in $B_1 \cup B_2$. Then by Example (iv) following Definition 2.1 we may assume that $\#(C \setminus H) = 2$; otherwise exchange (A_1, A_2) and (B_1, B_2).

Let $a_1, a_2 \in E$ denote the elements with $A_i = (A_1 \cap A_2) \cup \{a_i\}$ for $i \in \{1, 2\}$. Then one has $B_j = (B_1 \cap B_2) \cup \{a_i\}$ for $\{i, j\} = \{1, 2\}$.

Assume first that $B_1 \cap B_2 \subseteq H$. Then there exist $a \in (A_1 \cap A_2) \setminus (B_1 \cap B_2)$ and $b \in (B_1 \cap B_2) \setminus (A_1 \cap A_2)$ such that $((A_1 \cap A_2) \setminus \{a\}) \cup \{b\}$ generates the hyperplane H. Moreover, $A'_1 := (A_1 \setminus \{a\}) \cup \{b\}$ and $A'_2 := (A_2 \setminus \{a\}) \cup \{b\}$ are bases of M, because $a_1, a_2 \in E \setminus H$. Since $A'_1 \cap A'_2 \subseteq H$, the couple $((A'_1, A'_2), (B_1, B_2))$ is degenerate. But the couple $((A_1, A_2), (A'_2, A'_1))$ is also degenerate, because $(A_1 \cap A_2) \cup \{b\}$ is not a base of M. Thus in this case the induction step follows from $d(A'_1, B_1) = d(A'_2, B_2) = h - 1$ and

$$X_{A_1, A_2} \cdot X_{B_1, B_2} = \left(X_{A_1, A_2} \cdot X_{A'_2, A'_1}\right) \cdot \left(X_{A'_1, A'_2} \cdot X_{A'_2, A'_1}\right)^{-1} \cdot \left(X_{A'_1, A'_2} \cdot X_{B_1, B_2}\right).$$

In the remaining case choose some $b \in (B_1 \cap B_2) \setminus H$. Since $B_1 \Delta B_2 = \{a_1, a_2\} \subseteq C \setminus H$ and $\#(C \setminus H) = 2$, we must have $b \notin C$. Then there exists some $a \in A_1 \cap A_2$ such that $B'_1 := (B_1 \setminus \{b\}) \cup \{a\}$ and $B'_2 := (B_2 \setminus \{b\}) \cup \{a\}$ are bases of M. The circuit C is contained in $B'_1 \cup B'_2$. Therefore the couple $((A_1, A_2), (B'_1, B'_2))$ is degenerate. Moreover, the couple $((B'_2, B'_1), (B_1, B_2))$ is degenerate, too, because $(B_1 \cup B_2) \setminus \{b\}$ is not a base of M. Since $d(A_1, B'_1) = d(A_2, B'_2) = h - 1$, the induction step follows now similarly as in the first case. Thus we have verified that Definition 2.5 recovers [DW1, Definition 1.1].

Now we study the relations between the above definitions.

PROPOSITION 2.6. *The homomorphism* $\alpha : \mathbb{F}_M^{[\mathscr{F}]} \to \mathbb{T}_M^{\mathscr{F}}$ *defined by*

$$\alpha(\varepsilon) := \varepsilon_M^{\mathscr{F}},$$
$$\alpha\left(X_{\{e_1, \dots, e_m\}}\right) := T_{(e_1, \dots, e_m)}$$
$$\textit{for } \{e_1, \dots, e_m\} \in \mathscr{F},\ e_i < e_j \textit{ for } 1 \le i < j \le m$$

induces an isomorphism $\overline{\alpha} : \mathbb{T}_M^{[\mathscr{F}]} \hookrightarrow \mathbb{T}_M^{\mathscr{F}}$. *In particular, the structure of the group* $\mathbb{T}_M^{[\mathscr{F}]}$ *does not depend on the chosen total order "* $\le$ *" on* E.

Proof. Assume $e_1, \dots, e_k, f_1, \dots, f_l, a, b \in E$ with $e_1 < \cdots < e_k$, $f_1 < \cdots \lambda f_l$, $a < b$ and $\{a, b\} \cap \{e_i, f_j\} = \emptyset$ for all possible choices of i and j. Put

$$n_1 := \#\{i \mid a < e_i < b\}, \qquad n_2 := \#\{j \mid a < f_j < b\}.$$

If

$$((\{e_1, \dots, e_k, a\}, \{e_1, \dots, e_k, b\}), (\{f_1, \dots, f_l, b\}, \{f_1, \dots, f_l, a\})),$$

or

$$((\{e_1, \dots, e_k, a, b\}, \{e_1, \dots, e_k\}), (\{f_1, \dots, f_l\}, \{f_1, \dots, f_l, a, b\}))$$

is a degenerate couple in Γ_M then one has

$$\begin{aligned} &\alpha\Big(X_{\{e_1, \dots, e_k, a\}} \cdot X_{\{e_1, \dots, e_k, b\}}^{-1} \cdot \varepsilon^{n_1} \\ &\quad \times X_{\{f_1, \dots, f_l, b\}} \cdot X_{\{f_1, \dots, f_l, a\}}^{-1} \cdot \varepsilon^{n_2}\Big) \\ &\qquad = T_{(e_1, \dots, e_k, a)} \cdot T_{(e_1, \dots, e_k, b)}^{-1} \cdot T_{(f_1, \dots, f_l, b)} \cdot T_{(f_1, \dots, f_l, a)}^{-1} \\ &\qquad = 1 \end{aligned}$$

or

$$\begin{aligned} &\alpha\Big(X_{\{e_1, \dots, e_k, a, b\}} \cdot X_{\{e_1, \dots, e_k\}}^{-1} \cdot \varepsilon^{n_1} \\ &\quad \times X_{\{f_1, \dots, f_l\}} \cdot X_{\{f_1, \dots, f_l, a, b\}}^{-1} \cdot \varepsilon^{n_2}\Big) \\ &\qquad = T_{(e_1, \dots, e_k, a, b)} \cdot T_{(e_1, \dots, e_k)}^{-1} \cdot T_{(f_1, \dots, f_l)} \cdot T_{(f_1, \dots, f_l, a, b)}^{-1} \\ &\qquad = 1, \end{aligned}$$

respectively.

Thus α induces a homomorphism $\overline{\alpha} : \mathbb{T}_M^{[\mathscr{F}]} \to \mathbb{T}_M^{\mathscr{F}}$ which is trivially surjective. $\overline{\alpha}$ is even injective, because a similar computation as above shows that we have a well-defined homomorphism $\alpha' : \mathbb{T}_M^{\mathscr{F}} \to \mathbb{T}_M^{[\mathscr{F}]}$ given by

$$\begin{gathered}\alpha'\left(\varepsilon_M^{\mathscr{F}}\right) := \varepsilon_M^{[\mathscr{F}]},\\ \alpha'\left(T_{(e_1,\ldots,e_m)}\right) := T_{\{e_1,\ldots,e_m\}}\\ \text{for } \{e_1,\ldots,e_m\} \in \mathscr{F},\ e_i < e_j \text{ for } 1 \le i < j \le m,\end{gathered}$$

and one has $(\alpha' \circ \bar{\alpha})(T) = T$ for all $T \in \mathbb{T}_M^{[\mathscr{F}]}$. ∎

PROPOSITION 2.7. *The homomorphism $\Phi : \mathbb{F}_M \to \mathbb{T}_M^{[\mathscr{F}]}$ defined by*

$$\begin{gathered}\Phi(\varepsilon) := \varepsilon_M^{[\mathscr{F}]},\\ \Phi(X_{A,B}) := T_A \cdot T_B^{-1} \cdot \varepsilon_M(\{A,B\}) \qquad \textit{for } (A,B) \in \mathscr{F}^{(2)}\end{gathered}$$

induces a monomorphism $\overline{\Phi} : \mathbb{T}_M \hookrightarrow \mathbb{T}_M^{[\mathscr{F}]}$. If, furthermore, $\beta : \mathbb{T}_M^{[\mathscr{F}]} \twoheadrightarrow \mathbb{Z}$ is the obviously well-defined epimorphism given by

$$\beta\left(\varepsilon_M^{[\mathscr{F}]}\right) := 0, \qquad \beta(T_B) := 1 \qquad \textit{for } B \in \mathscr{F}$$

then the following sequence of abelian groups is exact:

$$0 \hookrightarrow \mathbb{T}_M \overset{\overline{\Phi}}{\hookrightarrow} \mathbb{T}_M^{[\mathscr{F}]} \overset{\beta}{\twoheadrightarrow} \mathbb{Z} \twoheadrightarrow 0. \tag{2.6}$$

In particular, one has

$$\mathbb{T}_M^{[\mathscr{F}]} \cong \mathbb{T}_M \times \mathbb{Z}. \tag{2.6a}$$

Proof. Clearly, Ker β is generated by $\varepsilon_M^{[\mathscr{F}]}$ and all products $T_A \cdot T_B^{-1}$, for which $A, B \in \mathscr{F}$. Since the base graph Γ_M is connected, it follows from the definition of Φ that $\Phi(\mathbb{F}_M) = \operatorname{Ker}\beta$. The rest of the theorem follows once it is shown that $\operatorname{Ker}\Phi = \mathbb{K}_M$.

First we verify that $\Phi(\mathbb{K}_M) = \{1\}$. Clearly, one has $\Phi(\varepsilon^2) = (\varepsilon_M^{[\mathscr{F}]})^2 = 1$ and

$$\Phi(X_{A,B} \cdot X_{B,A}) = \left(\varepsilon_M(\{A,B\})\right)^2 = 1 \qquad \text{for all } (A,B) \in \mathscr{F}^{(2)}.$$

Now assume $\{A_1, A_2, A_3\} \in \mathscr{D}$. By Lemma 2.4 there exist $A \in \mathscr{P}(E)$ and pairwise distinct $a_1, a_2, a_3 \in E$ with $A_j = A \Delta \{a_j\}$ for $1 \le j \le 3$. By symmetry we may assume $a_1 < a_2 < a_3$. Then we get

$$\varepsilon(\{A_1, A_3\}) = \begin{cases} \varepsilon(\{A_1, A_2\}) \cdot \varepsilon(\{A_2, A_3\}) & \text{if } a_2 \notin A \\ \varepsilon \cdot \varepsilon(\{A_1, A_2\}) \cdot \varepsilon(\{A_2, A_3\}) & \text{if } a_2 \in A \end{cases}$$

and thus

$$\Phi\left(X_{A_1,A_2}\cdot X_{A_2,A_3}\cdot X_{A_3,A_1}\right)=\begin{cases}1 & \text{for } \{A_1,A_2,A_3\}\in\mathscr{D}_1\\ \varepsilon_M^{[\mathscr{F}]} & \text{for } \{A_1,A_2,A_3\}\in\mathscr{D}_2,\end{cases}$$

as claimed.

If $((A,B),(A',B'))$ is a degenerate couple in Γ_M then one has

$$\begin{aligned}&\Phi(X_{A,B}\cdot X_{A',B'})\\&\quad=T_A\cdot T_B^{-1}\cdot\varepsilon_M(\{A,B\})\cdot T_{A'}\cdot T_{B'}^{-1}\cdot\varepsilon_M(\{A',B'\})\\&\quad=1.\end{aligned}$$

Thus it is proved that $\Phi(\mathbb{K}_M)=\{1\}$.

To show that the induced homomorphism $\Phi_0=\overline{\Phi}:\mathbb{T}_M\to\mathbb{T}_M^{[\mathscr{F}]}$ is injective, we construct a homomorphism $\omega_0:\mathbb{T}_M^{[\mathscr{F}]}\to\mathbb{T}_M$ with $\omega_0(\Phi_0(T))=T$ for all $T\in\mathbb{T}_M$.

For $\{A,B\}\in\mathscr{K}$ define $\operatorname{sign}_M(\{A,B\})\in\mathbb{T}_M$ by

$$\operatorname{sign}_M(\{A,B\}):=\begin{cases}1 & \text{for } \varepsilon_M(\{A,B\})=1\\ \varepsilon_M & \text{for } \varepsilon_M(\{A,B\})=\varepsilon_M^{[\mathscr{F}]}.\end{cases}$$

Now consider the—obviously well-defined—homomorphism $\gamma:C_1(\Gamma_M)\to\mathbb{T}_M$ given by

$$\gamma\left(\overline{(A,B)}\right):=T_{A,B}\cdot\operatorname{sign}_M(\{A,B\}).$$

We use Theorem 1.12 to show that $H_1(\Gamma_M)\subseteq\operatorname{Ker}\gamma$.

If $Z=(A_1,A_2,A_3,A_1)$ is a reentrant path in Γ_M of length 3, then one has

$$\begin{aligned}&\operatorname{sign}_M(\{A_1,A_2\})\cdot\operatorname{sign}_M(\{A_2,A_3\})\cdot\operatorname{sign}_M(\{A_3,A_1\})\\&\quad=\begin{cases}1 & \text{for } \{A_1,A_2,A_3\}\in\mathscr{D}_1\\ \varepsilon_M & \text{for } \{A_1,A_2,A_3\}\in\mathscr{D}_2\end{cases}\end{aligned}$$

and, thus,

$$\begin{aligned}&\gamma\left(\overline{(A_1,A_2)}+\overline{(A_2,A_3)}+\overline{(A_3,A_1)}\right)\\&\quad=T_{A_1,A_2}\cdot T_{A_2,A_3}\cdot T_{A_3,A_1}\\&\qquad\times\operatorname{sign}_M(\{A_1,A_2\})\cdot\operatorname{sign}_M(\{A_2,A_3\})\cdot\operatorname{sign}_M(\{A_3,A_1\})\\&\quad=1\end{aligned}$$

in any case.

If $Z = (A_1, A_2, A_3, A_4, A_1)$ is a strongly degenerate path in Γ_M of length 4, then $((A_1, A_2), (A_3, A_4))$ and $((A_2, A_3), (A_4, A_1))$ are degenerate couples in Γ_M. Moreover, there exist pairwise distinct $a_1, a_2, a_3, a_4 \in E$ with $a_1 < a_2$; $a_3 < a_4$ and

$$A_2 = A_1 \Delta \{a_1, a_2\}, \qquad A_3 = A_1 \Delta \{a_1, a_2, a_3, a_4\}, \qquad A_4 = A_1 \Delta \{a_3, a_4\}.$$

Thus we get by the definition of $\mathbb{T}_M$

$$\begin{aligned}
&\gamma\left(\overline{(A_1, A_2)} + \overline{(A_2, A_3)} + \overline{(A_3, A_4)} + \overline{(A_4, A_1)}\right) \\
&\quad = \left(T_{A_1, A_2} \cdot T_{A_3, A_4}\right) \cdot \left(T_{A_2, A_3} \cdot T_{A_4, A_1}\right) \\
&\qquad \times \left(\operatorname{sign}_M(\{A_1, A_2\}) \cdot \operatorname{sign}_M(\{A_3, A_4\})\right. \\
&\qquad\quad \left.\times \operatorname{sign}_M(\{A_2, A_3\}) \cdot \operatorname{sign}_M(\{A_4, A_1\})\right) \\
&\quad = 1^2 \cdot \varepsilon_M^{\#\{e \in A_1 \mid a_1 < e < a_2\} - \#\{e \in A_3 \mid a_1 < e < a_2\}} \\
&\qquad \times \varepsilon_M^{\#\{e \in A_2 \mid a_3 < e < a_4\} - \#\{e \in A_4 \mid a_3 < e < a_4\}} \\
&\quad = \varepsilon_M^{\#\{e \in \{a_3, a_4\} \mid a_1 < e < a_2\}} \cdot \varepsilon_M^{\#\{e \in \{a_1, a_2\} \mid a_3 < e < a_4\}} \\
&\quad = 1.
\end{aligned}$$

Now, Theorem 1.12 implies that $\gamma(H_1(\Gamma_M)) = \{1\}$; that means, for every reentrant path $(A_0, \ldots, A_m)$ in Γ_M one has

$$\prod_{i=1}^{m} \gamma\left(\overline{(A_i, A_{i-1})}\right) = 1.$$

If we thus fix some $F_0 \in \mathscr{F}$, we have a well-defined homomorphism $\omega : \mathbb{F}_M^{[\mathscr{F}]} \to \mathbb{T}_M$ given by

$$\omega(\varepsilon) := \varepsilon_M,$$

$$\omega\left(X_{F_0}\right) := 1,$$

$$\omega(X_F) := \prod_{i=1}^{l} \gamma\left(\overline{(F_i, F_{i-1})}\right),$$

where $F_l = F$ and $(F_0, \ldots, F_l)$ is some path in Γ_M from F_0 to F.

For $(A, B) \in \mathscr{F}^{(2)}$ one has

$$\omega\left(X_A \cdot X_B^{-1}\right) = \gamma\left(\overline{(A, B)}\right) = T_{A,B} \cdot \operatorname{sign}_M(\{A, B\}); \tag{2.7}$$

thus for a degenerate couple $((A_1, A_2), (B_1, B_2))$ in Γ_M one gets

$$\begin{aligned}
&\omega\left(X_{A_1} \cdot X_{A_2}^{-1} \cdot \varepsilon(\{A_1, A_2\}) \cdot X_{B_1} \cdot X_{B_2}^{-1} \cdot \varepsilon(\{B_1, B_2\})\right) \\
&\quad = T_{A_1, A_2} \cdot T_{B_1, B_2} = 1.
\end{aligned}$$

Since trivially $\omega(\varepsilon^2) = 1$, it turns out that $\omega(\mathbb{K}_M^{[\mathscr{F}]}) = \{1\}$; that is, ω induces a homomorphism $\omega_0 = \overline{\omega} : \mathbb{T}_M^{[\mathscr{F}]} \to \mathbb{T}_M$. By the definitions of Φ_0

and ω_0 and by (2.7) one gets

$$\omega_0\big(\Phi_0(\varepsilon_M)\big) = \omega_0\big(\varepsilon_M^{[\mathscr{F}]}\big) = \varepsilon_M$$

and

$$\omega_0\big(\Phi_0(T_{A,B})\big) = \omega_0\big(T_A \cdot T_B^{-1} \cdot \varepsilon_M(\{A,B\})\big) = T_{A,B}$$
$$\text{for } (A,B) \in \mathscr{F}^{(2)}.$$

Thus one has $\omega_0(\Phi_0(T)) = T$ for all $T \in \mathbb{T}_M$, which proves the injectivity of Φ_0, and the proposition follows. ∎

Note that the essential point in this proof is the fact that Maurer's homotopy theory holds for even Δ-matroids. To summarize the results of this section we state the following.

THEOREM 2.8. *Define the homomorphism* $\psi_0 : \mathbb{T}_M \to \mathbb{T}_M^{\mathscr{F}}$ *by*

$$\psi_0(T) := \overline{\alpha}\big(\overline{\Phi}(T)\big) \qquad \text{for } T \in \mathbb{T}_M,$$

and let $\beta_0 : \mathbb{T}_M^{\mathscr{F}} \to \mathbb{Z}$ *denote the epimorphism given by*

$$\beta_0\big(\varepsilon_M^{\mathscr{F}}\big) := 0,$$
$$\beta_0\big(T_{(e_1,\dots,e_m)}\big) := 1 \qquad \text{for } \{e_1,\dots,e_m\} \in \mathscr{F},\ e_i \neq e_j \text{ for } i \neq j.$$

Then one has

(i) *The following diagram commutes and has exact rows*:

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathbb{T}_M \arrow[r, hook, "\overline{\Phi}"] \arrow[d, two heads, "\mathrm{id}"] & \mathbb{T}_M^{[\mathcal{F}]} \arrow[r, two heads, "\beta"] \arrow[d, two heads, "\overline{\alpha}"] & \mathbb{Z} \arrow[r, two heads] \arrow[d, two heads, "\mathrm{id}"] & 0 \\
0 \arrow[r, hook] & \mathbb{T}_M \arrow[r, hook, "\psi_0"] & \mathbb{T}_M^{\mathcal{F}} \arrow[r, two heads, "\beta_0"] & \mathbb{Z} \arrow[r, two heads] & 0
\end{tikzcd}$$

(ii) *The monomorphism* ψ_0 *satisfies*

$$\psi_0(\varepsilon_M) = \varepsilon_M^{\mathscr{F}},$$
$$\psi_0\big(T_{\{e_1,\dots,e_m,a\},\{e_1,\dots,e_m,b\}}\big) = T_{(e_1,\dots,e_m,a)} \cdot T^{-1}_{(e_1,\dots,e_m,b)}$$
$$\text{for } \{e_1,\dots,e_m,a\} \in \mathscr{F},\ \{e_1,\dots,e_m,b\} \in \mathscr{F},$$
$$\psi_0\big(T_{\{e_1,\dots,e_m,a,b\},\{e_1,\dots,e_m\}}\big) = T_{(e_1,\dots,e_m,a,b)} \cdot T^{-1}_{(e_1,\dots,e_m)}$$
$$\text{for } \{e_1,\dots,e_m,a,b\} \in \mathscr{F},\ \{e_1,\dots,e_m\} \in \mathscr{F},\ a < b.$$

Proof. Both assertions follow trivially from Propositions 2.6 and 2.7. ∎

In the sequel we identify any $T \in \mathbb{T}_M$ with its images under the maps $\overline{\Phi}$ and ψ_0. In particular, from now on we write

$$\varepsilon_M = \varepsilon_M^{\mathscr{F}} = \varepsilon_M^{[\mathscr{F}]}, \tag{2.8a}$$

$$\varepsilon_M(\{A, B\}) = \operatorname{sign}_M(\{A, B\}) \qquad \text{for } \{A, B\} \in \mathscr{K}. \tag{2.8b}$$

EXAMPLE. Assume E is finite and $\mathscr{F} := \{F \subseteq E \mid \#F \equiv 0 \bmod 2\}$. Then the base graph Γ_M does not contain any degenerate couple. Therefore Definition 2.3 yields at once

$$\mathbb{T}_M^{[\mathscr{F}]} \cong (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^{\mathscr{F}} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z} & \text{for } E = \varnothing \\ (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^{(2^{\#E-1})}, & \text{otherwise}. \end{cases}$$

Thus Theorem 2.8 implies

$$\mathbb{T}_M^{\mathscr{F}} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z} & \text{for } E = \varnothing \\ (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^{(2^{\#E-1})}, & \text{otherwise}; \end{cases}$$

$$\mathbb{T}_M \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } E = \varnothing \\ (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^{(2^{\#E-1}-1)}, & \text{otherwise}. \end{cases}$$

3. THE TUTTE GROUP OF MINORS AND THE DUAL OF AN EVEN Δ-MATROID

As above let $M = (E, \mathscr{F})$ denote some even Δ-matroid. In this section we want to relate the group $\mathbb{T}_M^{\mathscr{F}}$ to $\mathbb{T}_{M'}^{\mathscr{F}'}$ for a minor $M' = (E', \mathscr{F}')$ of M and in case of finite E to $\mathbb{T}_{M^*}^{\mathscr{F}^*}$ for the dual M^* of M.

DEFINITION 3.1 (see also [BD, Section 2]). (i) Assume $E' \subseteq E$ such that

$$\mathscr{F}' := \{F \in \mathscr{F} \mid F \subseteq E'\} \tag{3.1a}$$

is nonempty. Then

$$M \setminus (E \setminus E') = M \mid E' := (E', \mathscr{F}') \tag{3.1b}$$

is called the *restriction of M to E'*.

(ii) Assume $I \subseteq E$ is finite such that

$$\mathscr{F}'' := \{F \setminus I \mid I \subseteq F,\, F \in \mathscr{F}\} \tag{3.2a}$$

is nonempty and put $E'' := E \setminus I$. Then

$$M/I = M.E'' := (E'', \mathscr{F}'') \tag{3.2b}$$

is called the *contraction of M to E''*.

First we prove the following.

LEMMA 3.2. (i) *Assume $M' = (E', \mathscr{F}')$ is the restriction of M to E' for some subset $E' \subseteq E$. Then a degenerate couple $((A_1, A_2), (A_3, A_4))$ in the base graph $\Gamma_{M'}$ of M' is also a degenerate couple in Γ_M.*

(ii) *Assume $M'' = (E'', \mathscr{F}'')$ is the contraction of M to $E'' = E \setminus I$ for some finite subset $I \subseteq E$. Suppose $((A_1, A_2), (A_3, A_4))$ is a degenerate couple in the base graph $\Gamma_{M''}$ of M'' and put $B_i := A_i \cup I$ for $1 \leq i \leq 4$. Then $((B_1, B_2), (B_3, B_4))$ is a degenerate couple in Γ_M.*

Proof. (i) follows from the fact that $A_1 \cup A_2 \cup A_3 \cup A_4$ and $E \setminus E'$ are disjoint; thus for $F \subseteq A_1 \cup A_2 \cup A_3 \cup A_4$ one has $F \in \mathscr{F}$ if and only if $F \in \mathscr{F}'$.

(ii) is also clear by Definition 2.1 (ii) and Definition 3.1 (ii), because $B_1 \cap B_2 \cap B_3 \cap B_4$ contains I, and the map $\sigma : E'' \to E$ defined by $\sigma(A) := A \cup I$ has the property that $A \in \mathscr{F}''$ holds if and only if $A \cup I \in \mathscr{F}$ is satisfied. ∎

Now we can show the following basic proposition.

PROPOSITION 3.3. (i) *Assume $M' = (E', \mathscr{F}')$ is the restriction of M to E' for some subset $E' \subseteq E$. For $\{a_1, \ldots, a_m\} \in \mathscr{F}'$ with $a_i \neq a_j$ for $i \neq j$ let $T'_{(a_1, \ldots, a_m)}$ denote the image of $X_{(a_1, \ldots, a_m)}$ in $\mathbb{T}_{M'}^{\mathscr{F}'}$. Then we have a well-defined homomorphism $\eta_1 : \mathbb{T}_{M'}^{\mathscr{F}'} \to \mathbb{T}_M^{\mathscr{F}'}$ given by*

$$\eta_1(\varepsilon_{M'}) := \varepsilon_M,$$

$$\eta_1\big(T'_{(a_1, \ldots, a_m)}\big) := T_{(a_1, \ldots, a_m)} \qquad \text{for } \{a_1, \ldots, a_m\} \in \mathscr{F}', \ a_i \neq a_j \text{ for } i \neq j.$$

(ii) *Assume $M'' = (E'', \mathscr{F}'')$ is the contraction of M to $E'' = E \setminus I$ for some finite subset $I = \{f_1, \ldots, f_l\}$ of E, $\#I = l$. For $\{a_1, \ldots, a_m\} \in \mathscr{F}''$ with $a_i \neq a_j$ for $i \neq j$ let $T''_{(a_1, \ldots, a_m)}$ denote the image of $X_{(a_1, \ldots, a_m)}$ in $\mathbb{T}_{M''}^{\mathscr{F}''}$. Then we have a well-defined homomorphism $\eta_2 : \mathbb{T}_{M''}^{\mathscr{F}''} \to \mathbb{T}_M^{\mathscr{F}}$ given by*

$$\eta_2(\varepsilon_{M''}) := \varepsilon_M,$$

$$\eta_2\big(T''_{(a_1, \ldots, a_m)}\big) := T_{(f_1, \ldots, f_l, a_1, \ldots, a_m)}$$

$$\text{for } \{a_1, \ldots, a_m\} \in \mathscr{F}'', \ a_i \neq a_j \text{ for } i \neq j.$$

Proof. (i) and (ii) follow directly from Lemma 3.2 (i), (ii), respectively. ∎

As an immediate consequence of Proposition 3.3 we obtain the following.

COROLLARY 3.4. *Assume M_0 is some minor of M, that is, a Δ-matroid, which is obtained by successively taking restrictions and contractions of M. Then the relation $\varepsilon_{M_0} = 1$ implies $\varepsilon_M = 1$.*

Remarks. (i) The homomorphism η_2 in Proposition 3.3 (ii) does, of course, depend on the labelling of the elements of I.

(ii) If we fix some total order on E, then the homomorphisms η_1, η_2 in Proposition 3.3 induce, in view of the results of Section 2, homomorphisms

$$\eta_1' : \mathbb{T}_{M'}^{[\mathscr{F}']} \to \mathbb{T}_M^{[\mathscr{F}]}, \qquad \eta_1'' : \mathbb{T}_{M'} \to \mathbb{T}_M,$$
$$\eta_2' : \mathbb{T}_{M''}^{[\mathscr{F}'']} \to \mathbb{T}_M^{[\mathscr{F}]}, \qquad \eta_2'' : \mathbb{T}_{M''} \to \mathbb{T}_M.$$

DEFINITION 3.5. Assume E is finite. Then the *dual* M^* of $M = (E, \mathscr{F})$ is defined by

$$M^* = (E, \mathscr{F}^*), \qquad \text{where } \mathscr{F}^* = \{E \setminus F \mid F \in \mathscr{F}\}.$$

If we fix some total order on E we get the following.

PROPOSITION 3.6. *Assume E is finite. Then the isomorphism $\eta_0 : \mathbb{F}_M \hookrightarrow \mathbb{F}_{M^*}$ defined by*

$$\eta_0(\varepsilon) := \varepsilon,$$
$$\eta_0(X_{A,B}) := \varepsilon \cdot X_{E \setminus A, E \setminus B} \qquad \textit{for } (A, B) \in \mathscr{F}^{(2)}$$

induces an isomorphism $\overline{\eta_0} : \mathbb{T}_M \hookrightarrow \mathbb{T}_{M^*}$.

Proof. If $\{i, j\} = \{1, 2\}$ and $\{A\Delta\{a_1\}, A\Delta\{a_2\}, A\Delta\{a_3\}\}$ is a triangle of the ith kind in Γ_M, then (2.4a) and (2.4b) imply that

$$\{(E \setminus A)\Delta\{a_1\}, (E \setminus A)\Delta\{a_2\}, (E \setminus A)\Delta\{a_3\}\}$$

is a triangle of the jth kind in Γ_{M^*}. Furthermore, $((A, B), (A', B'))$ is a degenerate couple in Γ_M if and only if

$$((E \setminus A, E \setminus B), (E \setminus A', E \setminus B'))$$

is a degenerate couple in Γ_{M^*}. Therefore, the result follows. ∎

4. THE TUTTE GROUP OF THE DIRECT SUM OF Δ-MATROIDS AND THE INNER TUTTE GROUP

Assume again that $M = (E, \mathscr{F})$ is an even Δ-matroid defined on some possibly infinite set E. In the sequel we want to study the connected components of M and compute the Tutte group of M from the Tutte groups of these components.

DEFINITION 4.1 (see [B4]). A subset E' of E is called a *separator* of $M = (E, \mathscr{F})$ if there exist set systems $\mathscr{F}' \subseteq \mathscr{P}(E')$ and $\mathscr{F}'' \subseteq \mathscr{P}(E'')$ for $E'' := E \setminus E'$ such that

$$\mathscr{F} = \left\{ A' \mathbin{\dot{\cup}} A'' \mid A' \in \mathscr{F}',\, A'' \in \mathscr{F}'' \right\}. \tag{4.1}$$

Note that (4.1) implies that $(E', \mathscr{F}')$ and $(E'', \mathscr{F}'')$ are even Δ-matroids and that $\mathscr{F}'$ and $\mathscr{F}''$ are uniquely determined by $\mathscr{F}$: If E' is a separator of M then with the above notations one has

$$\mathscr{F}' = \{ F \cap E' \mid F \in \mathscr{F} \}, \qquad \mathscr{F}'' = \{ F \cap E'' \mid F \in \mathscr{F} \}. \tag{4.1a}$$

The connected components of M will be even Δ-matroids defined on the minimal nonempty separators of M. First we want to derive an equivalent definition of these special separators.

DEFINITION 4.2. Two elements $a, b \in E$ are called *equivalent in M* if either $a = b$ or there exists some—finite—sequence $a_0, \dots, a_k$ in E such that $a_0 = a$, $a_k = b$, and for suitable $F_1, \dots, F_k, F_1', \dots, F_k' \in \mathscr{F}$ one has

$$F_i \Delta F_i' = \{ a_{i-1}, a_i \} \qquad \text{for } 1 \le i \le k.$$

In this case we write $a \sim b$. The equivalence classes determined by the equivalence relation "$\sim$" are called the *equivalence classes* of $M = (E, \mathscr{F})$.

PROPOSITION 4.3. *The equivalence classes of M are precisely the minimal nonempty separators of M.*

Proof. First we assume that E' is a minimal nonempty separator of M. We show that E' is the union of certain equivalence classes of M; that is, we must prove:

For $F_1, F_2 \in \mathscr{F}$ with $F_1 \Delta F_2 = \{e, f\}$, $e, f \in E$, one has either $\{e, f\} \subseteq E'$ or $\{e, f\} \cap E' = \emptyset$.

Put $E'' := E \setminus E'$. By assumption there exist set systems $\mathscr{F}' \subseteq \mathscr{P}(E')$ and $\mathscr{F}'' \subseteq \mathscr{P}(E'')$ such that

$$\mathscr{F} = \left\{ A' \mathbin{\dot{\cup}} A'' \mid A' \in \mathscr{F}',\, A'' \in \mathscr{F}'' \right\}.$$

Thus there exist $A_1', A_2' \in \mathscr{F}'$ and $A_1'', A_2'' \in \mathscr{F}''$ with $A_1' \dot\cup A_1'' = F_1$ and $A_2' \dot\cup A_2'' = F_2$. This means

$$A_1' \Delta A_2' = E' \cap \{e, f\}, \qquad A_1'' \Delta A_2'' = E'' \cap \{e, f\}.$$

Since $(E', \mathscr{F}')$ and $(E'', \mathscr{F}'')$ are even Δ-matroids we have

$$\#(A_1' \Delta A_2') \equiv \#(A_1'' \Delta A_2'') \equiv 0 \bmod 2,$$

and, therefore, the first part of the proposition follows.

Now assume that E' is an equivalence class of M. The result follows once it is shown that E' is also a separator of M.

Put $E'' := E \setminus E'$, and define $\mathscr{F}' \subseteq \mathscr{P}(E')$ and $\mathscr{F}'' \subseteq \mathscr{P}(E'')$ by

$$\mathscr{F}' := \{F \cap E' \mid F \in \mathscr{F}\}, \qquad \mathscr{F}'' := \{F \cap E'' \mid F \in \mathscr{F}\}.$$

Then it is clear that

$$\mathscr{F} \subseteq \{A' \dot\cup A'' \mid A' \in \mathscr{F}', A'' \in \mathscr{F}''\}.$$

Now assume that there exist $F', F'' \in \mathscr{F}$ with

$$(F' \cap E') \dot\cup (F'' \cap E'') \notin \mathscr{F}.$$

In view of $F'' = (F'' \cap E') \dot\cup (F'' \cap E'') \in \mathscr{F}$ choose some $F_1 \in \mathscr{F}$ with $F_0 := (F_1 \cap E') \dot\cup (F'' \cap E'') \in \mathscr{F}$ such that

$$d := \#\big((F' \Delta F_1) \cap E'\big) = \#\big((F' \Delta F_0) \cap E'\big)$$

is as small as possible.

Since $F_0 \cap E' = F_1 \cap E'$ we may assume $F_1 = F_0$; otherwise replace F_1 by F_0.

We have $d > 0$, because $(F' \cap E') \dot\cup (F'' \cap E'') \notin \mathscr{F}$; thus choose some $e \in (F' \Delta F_0) \cap E'$. Then there exists some $f \in (F' \Delta F_0) \setminus \{e\}$ with $F_0 \Delta \{e, f\} \in \mathscr{F}$. Since E' is an equivalence class of M we have $f \in E'$ and thus

$$F_0 \Delta \{e, f\} = \big((F_0 \Delta \{e, f\}) \cap E'\big) \dot\cup (F'' \cap E'') \in \mathscr{F}.$$

In view of $\#((F' \Delta F_0 \Delta \{e, f\}) \cap E') = d - 2 < d$ this contradicts the choice of F_1 and F_0; thus we have

$$\mathscr{F} = \{A' \dot\cup A'' \mid A' \in \mathscr{F}', A'' \in \mathscr{F}''\},$$

as claimed. ∎

Since by Definition 1.1 any $F \in \mathscr{F}$ is finite it is clear that there are only finitely many minimal nonempty separators of M which intersect at least one $F \in \mathscr{F}$. Any other minimal nonempty separator E' of M satisfies $\#E' = 1$ and $E' \cap F = \emptyset$ for all $F \in \mathscr{F}$.

DEFINITION 4.4 (see also [B4]). Assume E' is a minimal nonempty separator of M and put

$$\mathscr{F}' := \{F \cap E' \mid F \in \mathscr{F}\}.$$

Then the Δ-matroid $M' = (E', \mathscr{F}')$ is called a *connected component* of M.

Remark. Any connected component of M is a minor of M: If $M' = (E', \mathscr{F}')$ is as in Definition 4.4 and $F_0 \in \mathscr{F}$ is arbitrary then for $I := F_0 \setminus E'$ one has $M' = (M/I) \mid E'$.

Next we turn to the direct sum of even Δ-matroids.

DEFINITION 4.5. Assume $M_1 = (E_1, \mathscr{F}_1), \ldots, M_k = (E_k, \mathscr{F}_k)$ are even Δ-matroids defined on pairwise disjoint sets $E_1, \ldots, E_k$. Put

$$E_0 := \bigcup_{i=1}^{k} E_i, \qquad \mathscr{F}_0 := \left\{ \bigcup_{i=1}^{k} F_i \mid F_i \in \mathscr{F}_i \text{ for } 1 \le i \le k \right\}.$$

Then the even Δ-matroid

$$M_0 := M_1 + \cdots + M_k := (E_0, \mathscr{F}_0)$$

is called the *direct sum* of $M_1, \ldots, M_k$.

Remark. By the above definitions it is clear that any even Δ-matroid with finitely many connected components is the direct sum of these components.

Now we are able to compute the Tutte group of a direct sum of even Δ-matroids.

PROPOSITION 4.6. *Assume $M_1 = (E_1, \mathscr{F}_1), \ldots, M_k = (E_k, \mathscr{F}_k)$ and $M_0 = M_1 + \cdots + M_k = (E_0, \mathscr{F}_0)$ are as in Definition* 4.5. *Let "$\le$" denote some total order on E_0. Moreover, let $\mathbb{T}_0$ denote the subgroup of the direct product $\prod_{i=1}^{k} \mathbb{T}_{M_i}$ generated by all products $\varepsilon_{M_i} \cdot \varepsilon_{M_j}$ for $1 \le i < j \le k$. Then one has*

$$\mathbb{T}_{M_0} \cong \left(\prod_{i=1}^{k} \mathbb{T}_{M_i} \right) \Big/ \mathbb{T}_0; \tag{4.2}$$

that is, one gets $\mathbb{T}_{M_0}$ by taking the direct product of all $\mathbb{T}_{M_i}$, $1 \le i \le k$, and then identifying the elements ε_{M_i} for $1 \le i \le k$.

Proof. Put $\mathbb{T} := \prod_{i=1}^{k} \mathbb{T}_{M_i}$. First note that it is trivial that all E_i, $1 \le i \le k$, are separators of M_0.

For $1 \le i \le k$ and $(A_1, A_2) \in \mathscr{F}_i^{(2)}$ let $T_{A_1, A_2}^{(i)}$ denote the image of X_{A_1, A_2} in $\mathbb{T}_{M_i}$; for $(A_1, A_2) \in \mathscr{F}_0^{(2)}$ let T_{A_1, A_2} denote the image of X_{A_1, A_2} in $\mathbb{T}_{M_0}$. Choose some fixed $F_0 \in \mathscr{F}_0$ and put $F_i := F_0 \cap E_i$, $G_i := F_0 \setminus E_i$ for $1 \le i \le k$.

If $((A_1, A_2), (A_3, A_4))$ is a degenerate couple in the base graph Γ_{M_i} for some i, $1 \le i \le k$, then by Definition 4.5 it is clear that

$$((A_1 \cup G_i, A_2 \cup G_i), (A_3 \cup G_i, A_4 \cup G_i))$$

is a degenerate couple in Γ_{M_0}. If, moreover, $j \in \{1, 2\}$ and $\{A_1, A_2, A_3\}$ is a triangle in Γ_{M_i} for some i, $1 \le i \le k$, of the jth kind, then $\{A_1 \cup G_i, A_2 \cup G_i, A_3 \cup G_i\}$ is a triangle in Γ_{M_0} of the jth kind. Thus we get a well-defined homomorphism $\eta : \mathbb{T} \to \mathbb{T}_{M_0}$ by

$$\eta(\varepsilon_{M_i}) := \varepsilon_{M_0} \qquad \text{for } 1 \le i \le k, \tag{4.3a}$$

$$\eta\left(T_{A_1, A_2}^{(i)}\right) := T_{A_1 \cup G_i, A_2 \cup G_i} \qquad \text{for } (A_1, A_2) \in \mathscr{F}_i^{(2)}, 1 \le i \le k. \tag{4.3b}$$

η does not depend on the special choice of F_0, because for $F_0' \in \mathscr{F}_0$ and $G_i' := F_0' \setminus E_i \ne G_i$ the couple $((A_1 \cup G_i, A_2 \cup G_i), (A_2 \cup G_i', A_1 \cup G_i'))$ is degenerate in Γ_{M_0}. η is surjective, because for $(A, B) \in \mathscr{F}_0^{(2)}$ the set $A \Delta B$ is contained in some equivalence class of M_0, and thus by Proposition 4.3 one has $A \Delta B \subseteq E_i$ for some i with $1 \le i \le k$. This means

$$T_{A, B} = \eta\left(T_{A \cap E_i, B \cap E_i}^{(i)}\right).$$

It remains to show that $\operatorname{Ker} \eta = \mathbb{T}_0$. Clearly, one has $\mathbb{T}_0 \subseteq \operatorname{Ker} \eta$ by (4.3a). To show that $\operatorname{Ker} \eta \subseteq \mathbb{T}_0$ we construct a homomorphism $\omega : \mathbb{T}_{M_0} \to \mathbb{T}/\mathbb{T}_0$ by

$$\omega(\varepsilon_{M_0}) := \varepsilon_{M_i} \cdot \mathbb{T}_0 \qquad \text{for } 1 \le i \le k, \tag{4.4a}$$

$$\omega(T_{A, B}) := T_{A \cap E_i, B \cap E_i}^{(i)} \cdot \mathbb{T}_0 \qquad \text{if } (A, B) \in \mathscr{F}_0^{(2)}, A \Delta B \subseteq E_i \text{ for some } i, 1 \le i \le k. \tag{4.4b}$$

ω is well defined, because for a degenerate couple $((A, B), (A', B'))$ in Γ_{M_0} one has $A \Delta B = A' \Delta B' \subseteq E_i$ for some i, $1 \le i \le k$, and in case

$$(A \cap E_i, B \cap E_i) \ne (B' \cap E_i, A' \cap E_i)$$

it follows that $((A \cap E_i, B \cap E_i), (A' \cap E_i, B' \cap E_i))$ is a degenerate couple in Γ_{M_i}. Moreover, if $\{A_1, A_2, A_3\}$ is a triangle in Γ_{M_0} of the jth kind for some $j \in \{1, 2\}$, then it follows easily from Lemma 2.4 that the sets $A_1 \Delta A_2$, $A_2 \Delta A_3$, and $A_1 \Delta A_3$ are contained in some E_i, $1 \le i \le k$. But then $\{A_1 \cap E_i, A_2 \cap E_i, A_3 \cap E_i\}$ is a triangle of the jth kind in Γ_{M_i}.

The homomorphism $\omega \circ \eta : \mathbb{T} \to \mathbb{T}/\mathbb{T}_0$ is the canonical epimorphism. In particular, one has $\operatorname{Ker} \eta \subseteq \mathbb{T}_0$. ∎

In the rest of this section we want to study some distinguished subgroup $\mathbb{T}_M^{(0)}$ of $\mathbb{T}_M$. To this end we fix some total order "$\le$" on E. For $A \subseteq E$ let

$\delta_A \in \mathbb{Z}^E$ denote the map defined by

$$\delta_A(e) := \begin{cases} 1 & \text{for } e \in A, \\ 0 & \text{for } e \notin A. \end{cases}$$

For $a \in E$ we write also δ_a, instead of $\delta_{\{a\}}$.

DEFINITION 4.7. Let $\Theta : \mathbb{T}_M^{[\mathscr{F}]} \to \mathbb{Z}_{\text{fin}}^E \times \mathbb{Z}$ denote the—obviously well-defined—homomorphism given by[2]

$$\Theta(\varepsilon_M) := 0, \tag{4.5a}$$

$$\Theta(T_A) := (\delta_A, 1) \qquad \text{for } A \in \mathscr{F}. \tag{4.5b}$$

Then $\mathbb{T}_M^{(0)} := \operatorname{Ker} \Theta$ is called the *inner Tutte group* of M.

By Proposition 2.7 it is trivial that $\mathbb{T}_M^{(0)}$ is contained in $\mathbb{T}_M$. If $S := \Theta(\mathbb{T}_M^{[\mathscr{F}]})$ then we have the following exact sequence:

$$0 \hookrightarrow \mathbb{T}_M^{(0)} \hookrightarrow \mathbb{T}_M^{[\mathscr{F}]} \overset{\Theta}{\twoheadrightarrow} S \twoheadrightarrow 0. \tag{4.6}$$

Since S is free abelian, we get

$$\mathbb{T}_M^{[\mathscr{F}]} \cong \mathbb{T}_M^{(0)} \times S. \tag{4.6a}$$

In case of finite E we want to compute the rank of S. To this end we shall have to refine the concept of equivalence classes of M; this will now be done for arbitrary E.

DEFINITION 4.8. Assume E' is an equivalence class of the even Δ-matroid $M = (E, \mathscr{F})$.

(i) For $a, b \in E'$ we write

$$a \mid b$$

if there exist $A, B \in \mathscr{F}$ with $a \in A \setminus B$, $b \in B \setminus A$, and $B = (A \setminus \{a\}) \cup \{b\}$.

(ii) For $a, b \in E'$ we write

$$a \leftrightarrow b$$

if there exist $A, B \in \mathscr{F}$ with $a, b \in A$ and $B = A \setminus \{a, b\}$.

(iii) For $a, b \in E'$ we write

$$a \nu b$$

if $a \mid b$ or $a \leftrightarrow b$.

[2] Recall that $\mathbb{Z}_{\text{fin}}^E := \langle \{\delta_e \mid e \in E\} \rangle$.

(iv) The equivalence class E' is called *unstable* if at least one of the following conditions holds:

(U1) There exist $a, b \in E'$ with $a \mid b$ and $a \leftrightarrow b$.

(U2) There exist $a_0, a_1, \ldots, a_k \in E'$ with $a_k = a_0$, $a_{i-1} \nu a_i$ for $1 \leq i \leq k$ and

$$\#\{i \mid 1 \leq i \leq k,\, a_{i-1} \leftrightarrow a_i\} \equiv 1 \bmod 2.$$

Otherwise E' is called *stable*.

(v) For $a, b \in E'$ we write

$$a \overset{R}{\sim} b$$

if E' is unstable or there exist $a_0, \ldots, a_k \in E'$ with $a_0 = a$, $a_k = b$, $a_{i-1} \nu a_i$ for $1 \leq i \leq k$ and

$$\#\{i \mid 1 \leq i \leq k,\, a_{i-1} \leftrightarrow a_i\} \equiv 0 \bmod 2.$$

(vi) For $a, b \in E'$ we write

$$a \overset{\bar{R}}{\sim} b$$

if E' is unstable or the relation $a \overset{R}{\sim} b$ does not hold.

(vii) If E' is stable then for $a \in E'$ we put

$$E'(a) := \left\{b \in E' \mid a \overset{R}{\sim} b\right\}.$$

Remarks. (i) The Δ-matroid $M = (E, \mathscr{F})$ is a matroid (with base set $\mathscr{F}$) if and only if there do not exist any $a, b \in E$ with $a \leftrightarrow b$. In particular, any equivalence class of a matroid is stable.

(ii) For $a, b \in E$ one has $a \sim b$ if and only if $a \overset{R}{\sim} b$ or $a \overset{\bar{R}}{\sim} b$.

(iii) The relation “$\overset{R}{\sim}$” is an equivalence relation; in general, the relation “$\overset{\bar{R}}{\sim}$” is not an equivalence relation. For $a, b, c \in E$ with $a \overset{\bar{R}}{\sim} b \overset{\bar{R}}{\sim} c$ one has $a \overset{R}{\sim} c$.

(iv) If E' is stable, then for $a \in E'$ one has $E'(a) = E'$ or for all $b \in E' \setminus E'(a)$ one has

$$E' = E'(a) \mathbin{\dot{\cup}} E'(b).$$

EXAMPLE. Assume E is arbitrary and

$$\mathscr{F} := \{F \subseteq E \mid \#F \equiv 0 \bmod 2\}.$$

Then $M = (E, \mathscr{F})$ is connected; that is, E is the only equivalence class of M. If $\# E \geq 3$, then E is unstable.

We have the following rather simple but quite useful lemma.

LEMMA 4.9. *Assume $a, b \in E$. Then one has*

(i) $a \,|\, b$ *implies* $\delta_a - \delta_b \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

(ii) $a \leftrightarrow b$ *implies* $\delta_a + \delta_b \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

(iii) $a \overset{R}{\sim} b$ *implies* $\delta_a - \delta_b \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

(iv) $a \overline{\overset{R}{\sim}} b$ *implies* $\delta_a + \delta_b \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

(v) *If E' is some unstable equivalence class of M, then for $a \in E'$ one has* $2 \cdot \delta_a \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

Proof.

(i) Assume $A, B \in \mathscr{F}$ with $B = (A \setminus \{a\}) \cup \{b\}$, $a \in A \setminus B$, $b \in B \setminus A$. Then we get $\Theta(T_A \cdot T_B^{-1}) = \delta_a - \delta_b$.

(ii) For $A, B \in \mathscr{F}$, $a, b \in A$, and $B = A \setminus \{a, b\}$ one has $\Theta(T_A \cdot T_B^{-1}) = \delta_a + \delta_b$.

(iii), (iv) By induction it suffices to show

(I) $\delta_a - \delta_b \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$ and $b \,|\, c$ imply $\delta_a - \delta_c \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

(II) $\delta_a - \delta_b \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$ and $b \leftrightarrow c$ imply $\delta_a + \delta_c \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

(III) $\delta_a + \delta_b \in \Theta(\mathbb{T}_M^{[\mathscr{F}]}$ and $b \,|\, c$ imply $\delta_a + \delta_c \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

(IV) $\delta_a + \delta_b \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$ and $b \leftrightarrow c$ imply $\delta_a - \delta_c \in \Theta(\mathbb{T}_M^{[\mathscr{F}]})$.

These assertions follow directly from (i), (ii), and the equations

$$(\delta_a - \delta_b) + (\delta_b - \delta_c) = \delta_a - \delta_c,$$
$$(\delta_a - \delta_b) + (\delta_b + \delta_c) = \delta_a + \delta_c,$$
$$(\delta_a + \delta_b) - (\delta_b - \delta_c) = \delta_a + \delta_c,$$
$$(\delta_a + \delta_b) - (\delta_b + \delta_c) = \delta_a - \delta_c.$$

(v) follows immediately from (iv) by putting $b = a$. ∎

Put $\tilde{E} := \{e \in E \,|\, e \notin F$ for all $F \in \mathscr{F}\}$. Then it is clear that for $E' := E \setminus \tilde{E}$ and $M' := M \setminus \tilde{E} = M \,|\, E'$ one has $\mathbb{T}_M^{[\mathscr{F}]} = \mathbb{T}_{M'}^{[\mathscr{F}]}$ and $\mathbb{T}_M^{(0)} = \mathbb{T}_{M'}^{(0)}$. Thus in the rest of this section we assume that $\tilde{E} = \emptyset$ for simplicity. Then M contains only finitely many connected components.

Let $E_1, \ldots, E_{k'}$ denote the unstable equivalence classes and $E_{k'+1}, \ldots, E_{k'+k''}$ the stable equivalence classes of M. Put $k := k' + k''$.

Assume that $F_0 \in \mathscr{F}$ is fixed, say $F_0 = \emptyset$ if $\emptyset \in \mathscr{F}$. Moreover, define the homomorphism $\Theta_0 : \mathbb{T}_M^{[\mathscr{F}]} \to \mathbb{Z}_{\text{fin}}^E \times \mathbb{Z}$ by

$$\Theta_0(\varepsilon_M) := 0,$$

$$\Theta_0(T_F) := \left(\delta_F - \delta_{F_0}, 1\right) = \Theta\left(T_F \cdot T_{F_0}^{-1}\right) + (0,1).$$

We have the following.

LEMMA 4.10. *One has* $\operatorname{Ker}\Theta_0 = \operatorname{Ker}\Theta = \mathbb{T}_M^{(0)}$. *In particular,* $\operatorname{Ker}\Theta_0$ *does not depend on the choice of* F_0.

Proof. Define the linear isomorphism $\varphi : \mathbb{Z}_{\text{fin}}^E \times \mathbb{Z} \hookrightarrow \mathbb{Z}_{\text{fin}}^E \times \mathbb{Z}$ by

$$\varphi(\delta_e) := \delta_e \qquad \text{for } e \in E, \qquad \varphi(0,1) := \left(\delta_{F_0}, 1\right).$$

Then we have $\varphi \circ \Theta_0 = \Theta$ and thus $\operatorname{Ker}\Theta_0 = \operatorname{Ker}\Theta$. ∎

DEFINITION 4.11. Assume $1 \le i \le k$. If the equivalence class E_i is unstable (that means $1 \le i \le k'$) define the homomorphism $\gamma_i : \mathbb{Z}_{\text{fin}}^{E_i} \to \mathbb{Z}/2\mathbb{Z}$ by

$$\gamma_i(\delta_e) := \bar{1} := 1 + 2\mathbb{Z} \qquad \text{for } e \in E_i.$$

If the equivalence class E_i is stable (that means $k' < i \le k$) choose some fixed $e_i \in E_i$ and define the homomorphism $\gamma_i : \mathbb{Z}_{\text{fin}}^{E_i} \to \mathbb{Z}$ by

$$\gamma_i(\delta_e) := 1 \qquad \text{for } e \in E_i(e_i)$$

$$\gamma_i(\delta_e) := -1 \qquad \text{for } e \in E_i \setminus E_i(e_i).$$

Finally, define the homomorphism $\gamma_0 : \mathbb{Z}_{\text{fin}}^E \times \mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z})^{k'} \times \mathbb{Z}^{k''}$ by

$$\gamma_0\left(\sum_{i=1}^{k}\left(\sum_{e\in E_i} n_e \cdot \delta_e\right), l\right) := \left(\gamma_i\left(\sum_{e \in E_i} n_e \cdot \delta_e\right)\right)_{1 \le i \le k}.$$

We are now able to state and to prove the following.

THEOREM 4.12. *The following sequence of abelian groups is exact*:

$$0 \hookrightarrow \mathbb{T}_M^{(0)} \hookrightarrow \mathbb{T}_M^{[\mathscr{F}]} \xrightarrow{\Theta_0} \mathbb{Z}_{\text{fin}}^E \times \mathbb{Z} \overset{\gamma_0}{\twoheadrightarrow} (\mathbb{Z}/2\mathbb{Z})^{k'} \times \mathbb{Z}^{k''} \twoheadrightarrow 0. \tag{4.7}$$

In particular, for finite E *one has*

$$\mathbb{T}_M^{[\mathscr{F}]} \cong \mathbb{T}_M^{(0)} \times \mathbb{Z}^{\#E+1-k''}, \tag{4.7a}$$

$$\mathbb{T}_M \cong \mathbb{T}_M^{(0)} \times \mathbb{Z}^{\#E-k''}. \tag{4.7b}$$

Proof. Statement (4.7a) follows directly from the first assertion, while (4.7b) is an immediate consequence of (4.7a) and Proposition 2.7. It remains to show that the sequence in (4.7) is exact.

The homomorphism γ_0 is trivially surjective. In view of Lemma 4.10 the theorem follows, once it is shown that $\Theta_0(\mathbb{T}_M^{[\mathscr{F}]}) = \operatorname{Ker} \gamma_0$. For $F \in \mathscr{F}$ we verify by induction on $\#(F \Delta F_0)$:

$$\gamma_0\big(\Theta_0(T_F)\big) = 0.$$

In case $F = F_0$ one has $\gamma_0(\Theta_0(T_{F_0})) = \gamma_0((0,1)) = 0$. Now assume $F \neq F_0$ and choose $a, b \in F \Delta F_0$ with $F' := F\Delta\{a, b\} \in \mathscr{F}$. For some i with $1 \leq i \leq k$ we have $\{a, b\} \subseteq E_i$. Moreover, one has—whether E_i is stable or not:

$$\gamma_i(\delta_a) = \begin{cases} \gamma_i(\delta_b), & \text{if } \#F = \#F', \\ -\gamma_i(\delta_b), & \text{otherwise}. \end{cases}$$

In any case the induction hypothesis yields

$$\gamma_0\big(\Theta_0(T_F)\big) = \gamma_0\big(\Theta_0(T_{F'}) + (\delta_a, 0) - (\delta_a, 0)\big) = \gamma_0\big(\Theta_0(T_{F'})\big) = 0.$$

Finally, we prove by induction on $m := \sum_{e \in E} |\lambda_e| + |l|$:

For $v := (\sum_{e \in E} \lambda_e \cdot \delta_e, l) \in \operatorname{Ker} \gamma_0$ one has $v \in \Theta_0(\mathbb{T}_M^{[\mathscr{F}]})$.

In case $m = 0$ there is nothing to show.

In the general case we may assume $l = 0$, because for all $l \in \mathbb{Z}$ one has

$$(0, l) = \Theta_0\big(T_{F_0}^l\big) \in \Theta_0\big(\mathbb{T}_M^{[\mathscr{F}]}\big) \subseteq \operatorname{Ker} \gamma_0.$$

Let E_i denote some equivalence class of M such that $\lambda_a \neq 0$ holds for at least one $a \in E_i$. Put $v_i := \sum_{e \in E_i} \lambda_e \cdot \delta_e$. In view of $\gamma_i(v_i) = 0$ at least one of the following two cases occurs:

Case I. E_i is unstable and $|\lambda_a| \geq 2$.

Case II. There exists some $b \in E_i \setminus \{a\}$ with $\lambda_b \neq 0$ and

$$\gamma_i\big(\operatorname{sign} \lambda_a \cdot \delta_a + \operatorname{sign} \lambda_b \cdot \delta_b\big) = 0.$$

In Case I we get by Lemma 4.9 (v):

$$2 \cdot \delta_a \in \Theta_0\big(\mathbb{T}_M^{[\mathscr{F}]}\big) \subseteq \operatorname{Ker} \gamma_0.$$

(Note that $\Theta(\mathbb{T}_M^{[\mathscr{F}]}) \cap \mathbb{Z}_{\text{fin}}^E = \Theta_0(\mathbb{T}_M^{[\mathscr{F}]}) \cap \mathbb{Z}_{\text{fin}}^E$). This means

$$v' := \sum_{e \in E \setminus \{a\}} \lambda_e \cdot \delta_e + (\lambda_a - 2 \cdot \operatorname{sign} \lambda_a) \cdot \delta_a \in \operatorname{Ker} \gamma_0,$$

and thus the induction hypothesis implies $v' \in \Theta_0(\mathbb{T}_M^{[\mathscr{F}]})$. Therefore, one also has

$$v = v' + 2 \cdot \operatorname{sign} \lambda_a \cdot \delta_a \in \Theta_0\left(\mathbb{T}_M^{[\mathscr{F}]}\right).$$

Thus in the rest of the proof we assume that Case II holds. Then Lemma 4.9 (iii), (iv) and the definition of γ_i imply

$$\operatorname{sign} \lambda_a \cdot \delta_a + \operatorname{sign} \lambda_b \cdot \delta_b \in \Theta_0\left(\mathbb{T}_M^{[\mathscr{F}]}\right) \subseteq \operatorname{Ker} \gamma_0 .$$

Thus we have

$$\begin{aligned} v'' := \sum_{e \in E \setminus \{a, b\}} \lambda_e \cdot \delta_e + (\lambda_a - \operatorname{sign} \lambda_a) \cdot \delta_a \\ + (\lambda_b - \operatorname{sign} \lambda_b) \cdot \delta_b \in \operatorname{Ker} \gamma_0 . \end{aligned}$$

By the induction hypothesis we get $v'' \in \Theta_0(\mathbb{T}_M^{[\mathscr{F}]})$ and thus also $v \in \Theta_0(\mathbb{T}_M^{[\mathscr{F}]})$. ∎

Remark. Since each equivalence class of a matroid M is stable, Theorem 4.12 recovers [DW1, Theorem 1.5], where the corresponding result for the finite matroids is proved.

5. Δ-MATROIDS WITH COEFFICIENTS AND THEIR TUTTE GROUPS

In this section we want to study the relationship between the Tutte group of an even Δ-matroid and its representability. We first recall the definition of fuzzy rings which serve as coefficient domains for matroids and even Δ-matroids.

DEFINITION 5.1. A *fuzzy ring* $K = (K; +; \cdot; \varepsilon; K_0)$ consists of a set K, together with two compositions:

$$\text{“}+: K \times K \to K : (\kappa, \lambda) \mapsto \kappa + \lambda,\text{”}$$

$$\text{“}\cdot: K \times K \to K : (\kappa, \lambda) \mapsto \kappa \cdot \lambda,\text{”}$$

a specified element $\varepsilon \in K$, and a specified subset $K_0 \subseteq K$ such that the following axioms hold:

(FR0) $(K, +)$ and $(K, \cdot)$ are abelian semigroups with neutral elements 0 and 1, respectively;

(FR1) $0 \cdot \kappa = 0$ for all $\kappa \in K$;

(FR2) $\alpha \cdot (\kappa_1 + \kappa_2) = \alpha \cdot \kappa_1 + \alpha \cdot \kappa_2$ for all $\kappa_1, \kappa_2 \in K$ and $\alpha \in K^* := \{\beta \in K \mid 1 \in \beta \cdot K\}$, the group of *units* in K;

(FR3) $\varepsilon^2 = 1$;

(FR4) $K_0 + K_0 \subseteq K_0$, $K \cdot K_0 \subseteq K_0$, $0 \in K_0$, $1 \notin K_0$;

(FR5) for $\alpha \in K^*$ one has $1 + \alpha \in K_0$ if and only if $\alpha = \varepsilon$;

(FR6) $\kappa_1, \kappa_2, \lambda_1, \lambda_2 \in K$ and $\kappa_1 + \lambda_1, \kappa_2 + \lambda_2 \in K_0$ implies $\kappa_1 \cdot \kappa_2 + \varepsilon \cdot \lambda_1 \cdot \lambda_2 \in K_0$;

(FR7) $\kappa, \lambda, \kappa_1, \kappa_2 \in K$ and $\kappa + \lambda \cdot (\kappa_1 + \kappa_2) \in K_0$ implies $\kappa + \lambda \cdot \kappa_1 + \lambda \cdot \kappa_2 \in K_0$.

Remarks. (i) (FR4), (FR5), and (FR7) imply $\kappa + \varepsilon \cdot \kappa \in K_0$ for all $\kappa \in K$.

(ii) (FR4) yields directly $K^* \cap K_0 = \emptyset$.

(iii) (FR2), (FR4) and (FR5) imply

(FR5′) For $\alpha, \beta \in K^*$ one has $\alpha + \beta \in K_0$ if and only if $\beta = \varepsilon \cdot \alpha$.

EXAMPLES. (i) The commutative rings $R = (R; +; \cdot)$ with $1 \in R$ are (in a canonical correspondence to) exactly those fuzzy rings $(K; +; \cdot; \varepsilon; K_0)$ for which $K_0 = \{0\}$. In this case we have necessarily $\varepsilon = -1$.

(ii) A fuzzy ring K is a field if and only if $K^* = K \setminus \{0\}$.

(iii) If $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring and if $U \leq K^*$ is a subgroup of its group of units, then we can form the "quotient fuzzy ring"

$$K/U := \left(\mathscr{P}(K)^U; \dotplus; \cdot; \varepsilon \cdot U; \mathscr{P}(K)_0^U\right),$$

where $\mathscr{P}(K)^U$ denotes the nonempty U-invariant subsets of K (that is, $T \in \mathscr{P}(K)^U$ if and only if $U \cdot T = T \neq \emptyset$), which are added and multiplied as "complexes":

$$T_1 \dotplus T_2 := \{\kappa_1 + \kappa_2 \mid \kappa_1 \in T_1, \kappa_2 \in T_2\} \qquad \left(T_1, T_2 \in \mathscr{P}(K)^U\right),$$

and where $\mathscr{P}(K)_0^U$ denotes those U-invariant subsets $T \subseteq K$ with $T \cap K_0 \neq \emptyset$.

These and further examples of fuzzy rings are studied in more detail in [D, (1.3); W3, Section 1].

Now we extend the concept of a Δ-matroid with coefficients as introduced in [W3] to Δ-matroids defined on sets of arbitrary cardinality.

DEFINITION 5.2. Assume $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring and "$\leq$" is some fixed total order on some set E.

(i) A map $p : \mathscr{P}_{\text{fin}}(E) \to K^* \dot\cup \{0\}$ is a *twisted Pfaffian map*, if the following axioms hold:

(TP0) There exists some $I_0 \in \mathscr{P}_{\text{fin}}(E)$ with $p(I_0) \neq 0$.

(TP1) For all $I_1, I_2 \in \mathscr{P}_{\text{fin}}(E)$ with $p(I_1) \neq 0$ and $p(I_2) \neq 0$ one has $\# I_1 \equiv \# I_2 \bmod 2$.

(TP2) If $I_1, I_2 \in \mathscr{P}_{\mathrm{fin}}(E)$ and $I_1 \Delta I_2 = \{i_1, \dots, i_l\}$ with $i_j < i_{j+1}$ for $1 \le j \le l-1$, then one has

$$\sum_{j=1}^{l} \varepsilon^j \cdot p\big(I_1 \Delta \{i_j\}\big) \cdot p\big(I_2 \Delta \{i_j\}\big) \in K_0 .$$

Two twisted Pfaffian maps $p_1, p_2 : \mathscr{P}_{\mathrm{fin}}(E) \to K^* \dot{\cup} \{0\}$ are called *equivalent* if there exists some $\kappa \in K^*$ such that for all $I \in \mathscr{P}_{\mathrm{fin}}(E)$ one has $p_1(I) = \kappa \cdot p_2(I)$.

(ii) A *Δ-matroid M defined on E and with coefficients in K* consists of an equivalence class of twisted Pfaffian maps $p : \mathscr{P}_{\mathrm{fin}}(E) \to K^* \dot{\cup} \{0\}$. We write also $M = M_p$ for any twisted Pfaffian map p defining M. A subset $F \in \mathscr{P}_{\mathrm{fin}}(E)$ is called *free* or *feasible* in M if $p(F) \neq 0$ holds for one and thus for all twisted Pfaffian maps p with $M_p = M$.

Remarks. (i) If $\mathscr{F}$ is the set of feasible subsets of a Δ-matroid $M = M_p$ with coefficients in some fuzzy ring K, then the pair $\mathbf{M} := (E, p^{-1}(K^*)) = (E, \mathscr{F})$ is an even Δ-matroid defined on E. Indeed, (TP2) implies trivially the strong exchange condition. We call $\mathbf{M}$ the underlying Δ-matroid of $M = M_p$.

(ii) In [W3, Section 4] it is shown that the Δ-matroids with coefficients defined on any finite set E and equicardinal feasible sets are precisely the matroids with coefficients defined on E.

EXAMPLE. Assume K is a field and $(E, \mathscr{F})$ is a Δ-matroid defined on some finite set E. By definition (cf. [B2, Section 4]), $(E, \mathscr{F})$ is representable over K by some skew-symmetric matrix $A = (a_{ij})_{i,j \in E}$ with coefficients in K if there exists some $I \subseteq E$ such that (cf. (1.1))

$$\mathscr{F} \Delta I = \mathscr{F}(A) := \Big\{ F \subseteq E \mid A' := (a_{ij})_{i,j \in F} \text{ is nonsingular} \Big\}, \qquad (5.1)$$

where $(a_{ij})_{i,j \in \emptyset}$ is considered to be nonsingular.

In [W3, Theorem 2.9] it is shown that a Δ-matroid $(E, \mathscr{F})$ is representable by some skew-symmetric matrix if and only if $(E, \mathscr{F})$ is the underlying Δ-matroid of M_p for some twisted Pfaffian map $p : \mathscr{P}(E) \to K$. This result is a consequence of the fact that the map $p : \mathscr{P}(E) \to K$ defined in terms of the Pfaffian form Pf, that is $p(I) := Pf((a_{ij})_{i,j \in I})$ for $I \subseteq E$, satisfies

$$\sum_{j=1}^{l} (-1)^j \cdot p\big(I_1 \Delta \{i_j\}\big) \cdot p\big(I_2 \Delta \{i_j\}\big) = 0$$

for $I_1, I_2 \subseteq E$, $I_1 \Delta I_2 = \{i_1, \dots, i_l\}$, $i_1 < \cdots < i_l$ (see also [W2, Proposition 2.3]).

The next result relates a Δ-matroid with coefficients with its Tutte group.

PROPOSITION 5.3. *Assume $M = M_p$ is a Δ-matroid with coefficients in the fuzzy ring K defined on $(E, \leq)$ and $\mathbf{M} = (E, p^{-1}(K^*)) = (E, \mathscr{F})$ as its underlying Δ-matroid. Write $\mathbb{F}_M^{[\mathscr{F}]} := \mathbb{F}_{\mathbf{M}}^{[\mathscr{F}]}$ and $\mathbb{T}_M^{[\mathscr{F}]} := \mathbb{T}_{\mathbf{M}}^{[\mathscr{F}]}$. Then the homomorphism $\alpha_p : \mathbb{F}_M^{[\mathscr{F}]} \to K^*$ defined by*

$$\alpha_p(\varepsilon) := \varepsilon, \qquad \alpha_p(X_F) := p(F) \qquad \text{for } F \in \mathscr{F}$$

induces a homomorphism $\varphi_p = \overline{\alpha_p} : \mathbb{T}_M^{[\mathscr{F}]} \to K^*$.

Proof. Clearly, we have $\alpha_p(\varepsilon^2) = 1$. Now assume that $((A, B), (A', B'))$ is a degenerate couple in the base graph $\Gamma_M := \Gamma_{\mathbf{M}}$ of M. Then by Definition 2.1 (ii) there exist $a, b \in A \Delta A'$ with $B = A\Delta\{a, b\}$ and $B' = A'\Delta\{a, b\}$ such that for all $a' \in (A \Delta A') \setminus \{a, b\}$ one has

$$A\Delta\{a, a'\} \notin \mathscr{F} \qquad \text{or} \qquad A'\Delta\{a, a'\} \notin \mathscr{F}.$$

By (TP2) we get with $(A\Delta\{a\})\Delta(A'\Delta\{a\}) = A \Delta A' = \{x_1, \dots, x_m\}$, $x_1 < \cdots < x_m$, $\{a, b\} = \{x_i, x_j\}$, $i < j$:

$$\sum_{\nu=1}^{m} \varepsilon^{\nu} \cdot p\big(A\Delta\{a\}\Delta\{x_\nu\}\big) \cdot p\big(A'\Delta\{a\}\Delta\{x_\nu\}\big) \in K_0;$$

that is,

$$p(A) \cdot p(A') + \varepsilon^{j-i} \cdot p(B) \cdot p(B') \in K_0.$$

Thus (FR5′) implies

$$\begin{aligned}
&\alpha_p\big(X_A \cdot X_B^{-1} \cdot \varepsilon(\{A, B\}) \cdot X_{A'} \cdot X_{B'}^{-1} \cdot \varepsilon(\{A', B'\})\big) \\
&\quad = p(A) \cdot p(B)^{-1} \cdot p(A') \cdot p(B')^{-1} \\
&\qquad\quad \times \varepsilon^{\#\{e \in A \cap B \mid x_i < e < x_j\} + \#\{e \in A' \cap B' \mid x_i < e < x_j\}} \\
&\quad = p(A) \cdot p(B)^{-1} \cdot p(A') \cdot p(B')^{-1} \cdot \varepsilon^{\#\{e \in A \Delta A' \mid x_i < e < x_j\}} \\
&\quad = p(A) \cdot p(B)^{-1} \cdot p(A') \cdot p(B')^{-1} \cdot \varepsilon^{j-i-1} \\
&\quad = 1.
\end{aligned}$$

∎

Remark. In [W6, Section 4] it is shown that Proposition 5.3 becomes wrong if one replaces the Tutte group by some proper factor group.

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